A pure jump Markov process associated with Smoluchowski's coagulation equation

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A PURE JUMP MARKOV PROCESS ASSOCIATED WITH SMOLUCHOWSKI'S COAGULATION EQUATION

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The aim of the present paper is to construct a stochastic process, whose law is the solution of the Smoluchowski's coagulation equation. We introduce first a modified equation, dealing with the evolution of the distribution $Q_t(dx)$ of the mass in the system. The advantage we take on this is that we can perform an unified study for both continuous and discrete models.

The integro-partial-differential equation satisfied by $\{Q_t\}_{t\geq 0}$ can be interpreted as the evolution equation of the time marginals of a Markov pure jump process. At this end we introduce a nonlinear Poisson driven stochastic differential equation related to the Smoluchowski equation in the following way: if X_t satisfies this stochastic equation, then the law of X_t satisfies the modified Smoluchowski equation. The nonlinear process is richer than the Smoluchowski equation, since it provides historical information on the particles.

Existence, uniqueness and pathwise behavior for the solution of this SDE are studied. Finally, we prove that the nonlinear process X can be obtained as the limit of a Marcus–Lushnikov procedure.

1. Introduction. The coagulation model governs various phenomena as for example: polymerization, aggregation of colloidal particles, formation of stars and planets, behavior of fuel mixtures in engines, etc.

Smoluchowski's coagulation equation models the dynamic of such phenomena and describes the evolution of a system of clusters which coalesce in order to form bigger clusters. Each cluster is identified by its size. The only mechanism taken into account is the coalescence of two clusters, other effects as multiple coagulation are neglected. We assume also that the rate of these reactions depends on the sizes of clusters involved in the coagulation. Denoting by $n(k,t)$ the (nonnegative) concentration of clusters of size k at time t, the discrete

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Smoluchowski coagulation equation reads, for $k \in \mathbb{N}^*$:

$$
\text{(SD)} \qquad \begin{cases} \dfrac{d}{dt} n(k,t) = \dfrac{1}{2} \displaystyle\sum_{j=1}^{k-1} K(j,k-j) n(j,t) n(k-j,t) \\ \qquad\qquad\qquad - n(k,t) \displaystyle\sum_{j=1}^{\infty} K(j,k) n(j,t), \\ n(k,0) = n_0(k). \end{cases}
$$

The coagulation kernel K is naturally supposed to be nonnegative [i.e., $K : (\mathbb{N}^*)^2 \to \mathbb{R}_+$] and symmetric [i.e., $K(i,j) = K(j,i)$].

This system describes a nonlinear evolution equation of infinite dimension, with initial condition $(n_0(k))_{k\geq 1}$. In the first line of (SD), the terms on the right-hand side describes the creation of clusters of mass k by coagulation of clusters of mass j and $k-j$. This is the gain term. The coefficient $1/2$ is due to the fact that K is symmetric. The second term corresponds to the depletion of clusters of mass k after coalescence with other clusters. It represents the loss term.

The continuous analog of the equation (SD) can be written naturally:

$$
\text{(SC)} \qquad \begin{cases} \dfrac{\partial}{\partial t} n(x,t) = \dfrac{1}{2} \displaystyle\int_0^x K(y,x-y) n(y,t) n(x-y,t)\, dy \\ \qquad\qquad\qquad - n(x,t) \displaystyle\int_0^{\infty} K(x,y) n(y,t)\, dy, \\ n(x,0) = n_0(x) \end{cases}
$$

for all $x \in \mathbb{R}_+$. As above, the coagulation kernel K is nonnegative and symmetric. Existence and uniqueness results for these equations can be found, for example, in Ball and Carr [2] and Heilmann [10] (for the discrete subadditive case), Jeon [12] (for the discrete coagulation–fragmentation equation approached by Markov chains), Aldous [1] (for the continuous case) and Norris [16], [17] (for results generalizing to the continuous coagulation equation those of Jeon). We refer also to Deaconu and Tanré [4] for a probabilistic interpretation of the additive, multiplicative and constant kernels and for renormalization properties of the solution.

Our approach to (SC) or (SD) is new and purely stochastic. We construct a pure jump stochastic process $(X_t)_{t\geq 0}$ whose law is the solution of the Smoluchowski coagulation equation in the following sense: in the discrete case, $\mathbb{P}[X_t = k] = kn(k,t)$ for all $t \geq 0$ and all $k \in \mathbb{N}^*$, while in the continuous case, $\mathbb{P}[X_t \in dx] = xn(x,t)\, dx$ for all $t \geq 0$. For each ω, $X_t(\omega)$ has to be understood as the evolution of the size of a sort of a "typical" particle in the system.

The jump process satisfies a nonlinear Poisson driven stochastic differential equation. This nonlinear process is a richer structure than the solution of the

Smoluchowski equation, because it provides an historical information on the particle behavior.

This approach is strongly inspired by probabilistic works on Boltzmann equation. The Boltzmann equation deals with the distribution of the speeds in a gas, and can be related to the Smoluchowski equation for two reasons: first, it concerns the evolution of the "density of particles of speed v at time t," while the Smoluchowski equation deals with the "density of particles of mass x at time t." Second, the phenomena are discontinuous: in each case, a particle moves instantly from a mass x (or a speed v) to a new mass x' (or a speed v') after a coagulation (or a collision).

We refer to Tanaka [19], who introduced first a nonlinear jump process in order to study the Boltzmann equation of Maxwell molecules. Other results on this topic, based on probabilistic approach, were obtained by Desvillettes, Graham and Méléard [5], [9] or Fournier and Méléard [7], [6]. We follow essentially here the approach of [7] in which Tanaka's approach has been extended to the case of non-Maxwell molecules. The main fact that makes the Maxwell molecules easy to treat is that the rate of collision of a particle does not depend on its speed, which is not the case for non-Maxwell molecules. In the Smoluchowski's equation, the "rate of coagulation" of a typical particle depends on its size.

We get rid of this problem by using a sort of "reject" procedure: as in [7], there is, in our stochastic equation, an indicator function which allows to control the rate of coagulation.

Let us finally describe the plan of the present paper.

In Section 2, we introduce our notation and the modified Smoluchowski equation (MS), which allows us to study together equations (SC) and (SD). The equation (MS) describes the evolution of the distribution $Q_t(dx)$ (either discrete or continuous) of the sizes: for each t, Q_t is a probability measure on $\mathbb{R}_+^*$. Afterwards we relate (MS) to a nonlinear martingale problem (MP): for Q a solution to (MP), its time marginals Q_t satisfy the equation (MS). We finally exhibit a nonlinear Poisson driven stochastic differential equation (SDE), which gives a pathwise representation of (MP). If X_t satisfies (SDE), then its law is a solution to (MP). Notice that X_t can be seen as the evolution of a particle chosen randomly in the system, which coagulates randomly with other particles who are also chosen randomly. In other words, X_t is the evolution of the mass of a "typical" particle. In Section 3, we state and prove an existence result for (SDE), under quite general assumptions. The pathwise properties of the solution to (SDE) are briefly discussed in Section 4. Section 5 deals with uniqueness results for (SDE). In Section 6 we present the link of our process with the classical Marcus–Lushnikov process. The last section is the Appendix which includes some useful classical results.

A forthcoming paper will present a stochastic particle system associated with the process constructed in the present paper.

In the sequel A and B stand for constants whose values may change from line to line.

2. Framework. Our probabilistic approach is based on the following a priori remark: there is conservation of mass in (SC) and (SD). We expect in the discrete case that a solution $(n(k,t))_{t\geq 0, k\in\mathbb{N}^*}$ of (SD) should satisfy until a time $T_0 \leq \infty$,

$$\text{for all } t \in [0, T_0), \qquad \sum_{k\geq 1} k\, n(k,t) = 1 \tag{2.1}$$

and in the continuous one that a solution $(n(x,t))_{t\geq 0, x\in\mathbb{R}_+^*}$ of (SC) should satisfy until a time $T_0 \leq \infty$,

$$\text{for all } t \in [0, T_0), \qquad \int_0^\infty x\, n(x,t)\, dx = 1. \tag{2.2}$$

Thus, either in the discrete or continuous case, the quantity

$$Q_t(dx) = \sum_{k\geq 1} k\, n(k,t)\delta_k(dx) \quad \text{or} \quad Q_t(dx) = x\, n(x,t)\, dx \tag{2.3}$$

(where δ_k denotes the Dirac mass at k) is a probability measure on $\mathbb{R}_+$ for all $t \in [0, T_0)$.

For any t, $Q_t(dx)$ can be interpreted as the distribution of the mass of the particles at time t. We will rewrite equations (SD) and (SC) in terms of Q_t.

We begin with some notation.

NOTATION 2.1.

1. We denote by $C_b^1(\mathbb{R}_+)$ the set of bounded functions with a bounded and continuous derivative on $\mathbb{R}_+$.
2. We denote by $\mathcal{P}_1$ the set of probability measures Q on $\mathbb{R}_+^*$ such that

$$\int_{\mathbb{R}_+} x\, Q(dx) < \infty. \tag{2.4}$$

3. For $Q_0 \in \mathcal{P}_1$, we denote by

$$\mathcal{H}_{Q_0} = \overline{\left\{ \sum_{i=1}^n x_i ; x_i \in \operatorname{Supp} Q_0, n \in \mathbb{N}^* \right\}}^{\mathbb{R}_+}. \tag{2.5}$$

Notice that $\mathcal{H}_{Q_0}$ is a closed subset of $\mathbb{R}_+$ containing the support of Q_0. Since Q_0 is the distribution of the sizes of the particles in the initial system, $\mathcal{H}_{Q_0}$ simply represents the smallest closed subset of $\mathbb{R}_+$ in which the sizes of the particles will always take their values.

Also, the assumption $Q_0 \in \mathcal{P}_1$ simply means that the initial condition of the Smoluchowski equation admits a moment of order 2: in the discrete case this writes $\sum_k k^2 n_0(k) < \infty$, while in the continuous case we have, $\int_{\mathbb{R}_+} x^2 n_0(x)\, dx < \infty$.

DEFINITION 2.2. Let Q_0 be a probability measure on $\mathbb{R}_+$ belonging to $\mathcal{P}_1$ and let $T_0 \leq \infty$. We will say that a family $(Q_t(dx))_{t\in[0,T_0)}$ of probability measures on $\mathbb{R}_+$ is a weak solution to (MS) on $[0, T_0)$ with initial condition Q_0 if:

(a) for all $t \in [0, T_0)$, $\operatorname{Supp} Q_t \subset \mathcal{H}_{Q_0}$,
(b) for all $t \in [0, T_0)$, $\sup_{s\in[0,t]} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} K(x, y) Q_s(dx) Q_s(dy) < \infty$,
(c) and for all $\varphi \in C_b^1(\mathbb{R}_+)$ and all $t \in [0, T_0)$,

$$\begin{aligned}\int_{\mathbb{R}_+} \varphi(x) Q_t(dx) &\\ = \int_{\mathbb{R}_+} \varphi(x) Q_0(dx) &\\ + \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} [\varphi(x+y) - \varphi(x)] \frac{K(x, y)}{y} Q_s(dy) Q_s(dx)\, ds. &\end{aligned} \tag{2.6}$$

This definition allows us to treat together discrete and continuous cases. To make this assertion clear, let us state the following result:

PROPOSITION 2.3. *Let* $(Q_t(dx))_{t\in[0,T_0)}$ *be a weak solution to* (MS), *with initial condition* $Q_0 \in \mathcal{P}_1$, *for some* $T_0 \le \infty$.

(i) *If* $\operatorname{Supp} Q_0 \subset \mathbb{N}^*$, *then clearly* $\mathcal{H}_{Q_0} \subset \mathbb{N}^*$. *Thus for all* $t \in [0, T_0)$, $\operatorname{Supp} Q_t \subset \mathbb{N}^*$, *and we can write* Q_t *as*:

$$Q_t(dx) = \sum_{k\ge 1} \alpha_k(t) \delta_k(dx) \qquad \text{where } \alpha_k(t) = Q_t(\{k\}). \tag{2.7}$$

Then, *the function* $n(k, t) = \alpha_k(t)/k$ *is a solution to* (SD) *on* $[0, T_0)$ *with initial condition* $n_0(k) = \alpha_k(0)/k$, *in the following weak sense*; *for all* $t \in [0, T_0)$:

(a) $\sum_{k\ge1} kn(k, t) = 1$,
(b) $\sup_{s\in[0,t]} \sum_{k\ge1} \sum_{j\ge1} kjK(j, k)n(j, s)n(k, s) < \infty$,
(c) *and for all* $k \ge 1$,

$$\begin{aligned} n(k, t) = n_0(k) + \int_0^t \Bigg[\frac{1}{2} \sum_{i=1}^{k-1} n(i, s) n(k-i, s) K(i, k-i) &\\ - \sum_{j\ge1} n(k, s) n(j, s) K(k, j) \Bigg] ds. &\end{aligned} \tag{2.8}$$

(ii) *Assume now that*, *for all* $t \in [0, T_0)$, *the probability measure* Q_t *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}_+$ (*see Proposition* 5.3 *below*). *We can then write* $Q_0(dx) = f_0(x)\,dx$ *and*, *for any* $t \in (0, T_0)$, $Q_t(dx) = f(x, t)\,dx$. *Then* $n(x, t) = f(x, t)/x$ *is a solution to* (SC) *on* $[0, T_0)$ *with initial condition* $n_0(x) = f_0(x)/x$, *in the following weak sense*; *for all* $t \in [0, T_0)$:

(a) $\int_{\mathbb{R}_+} xn(x, t)\, dx = 1$,
(b) $\sup_{s\in[0,t]} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} xyK(x, y)n(x, s)n(x-y, s)\, dx\, dy < \infty$,

(c) *and for all test function* φ *such that* $\varphi(x)/x$ *belongs to* $C_b^1(\mathbb{R}_+^*)$,

$$\begin{aligned}\int_{\mathbb{R}_+}\varphi(x)n(x,t)\,dx &= \int_{\mathbb{R}_+}\varphi(x)n_0(x)\,dx \\ &\quad + \tfrac{1}{2}\int_0^t\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}[\varphi(x+y)-\varphi(x)-\varphi(y)] \\ &\qquad \times K(x,y)n(x,s)n(y,s)\,dx\,dy\,ds\end{aligned} \tag{2.9}$$

(*for a similar definition*, *see Norris* [16]).

(iii) *Other cases*, *as mixed cases*, *are contained in* (MS).

PROOF. First notice that in both cases, the integrability estimates on n are straightforward consequences of the integrability estimates on Q.

Step 1. Since $Q_t(dx) = \sum_{k\geq 1}\alpha_k(t)\delta_k(dx)$, with $\alpha_k(t) = kn(k,t)$, is a weak solution to (MS), we may apply (2.6) with $\varphi_k(x) \in C_b^1(\mathbb{R}_+^*)$ such that for some $k \geq 1$

$$\varphi_k(x) = \begin{cases} 0, & \text{if } x \notin \left[k-\dfrac{1}{2}, k+\dfrac{1}{2}\right], \\ \dfrac{1}{k}, & \text{if } x = k. \end{cases} \tag{2.10}$$

We obtain

$$\frac{\alpha_k(t)}{k} = \frac{\alpha_k(0)}{k} + \int_0^t \frac{1}{k}\sum_{i\geq 1}\alpha_i(s)\sum_{j\geq 1}\alpha_j(s)[\mathbb{1}_{\{i+j=k\}} - \mathbb{1}_{\{i=k\}}]\frac{K(i,j)}{j}\,ds \tag{2.11}$$

and thus

$$\begin{aligned} n(k,t) &= n_0(k) + \int_0^t\left[\sum_{i=1}^{k-1}\alpha_i(s)n(k-i,s)\frac{K(i,k-i)}{k}\right. \\ &\qquad\qquad \left. - \sum_{j\geq 1}n(k,s)n(j,s)K(k,j)\right]ds \\ &= n_0(k) + \int_0^t\left[\frac{1}{2}\sum_{i=1}^{k-1}n(i,s)n(k-i,s)K(i,k-i)\right. \\ &\qquad\qquad \left. - \sum_{j\geq 1}n(k,s)n(j,s)K(k,j)\right]ds \end{aligned} \tag{2.12}$$

where the last equality comes from the fact that $\alpha_i(s) = in(i,s)$ and $K(i,j)$ is a symmetric kernel.

Step 2. We now assume that $Q_t(dx) = f(x,t)\,dx$ for all $t \in [0,T_0)$; let φ be a test function such that $\psi(x) = \varphi(x)/x$ belongs to $C_b^1(\mathbb{R}_+)$. Applying (2.6) to ψ, we obtain, since K is symmetric,

$$
\begin{aligned}
\int_{\mathbb{R}_+} \varphi(x) n(x,t)\,dx
&= \int_{\mathbb{R}_+} \varphi(x) n_0(x)\,dx \\
&\quad + \int_0^t \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \left[\frac{\varphi(x+y)}{x+y} - \frac{\varphi(x)}{x}\right] \frac{K(x,y)}{y} f(x,s) f(y,s)\,dx\,dy\,ds \\
&= \int_{\mathbb{R}_+} \varphi(x) n_0(x)\,dx \\
&\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \left[\frac{\varphi(x+y)}{y(x+y)} - \frac{\varphi(x)}{xy} + \frac{\varphi(x+y)}{x(x+y)} - \frac{\varphi(y)}{xy}\right] \\
&\qquad\qquad \times xyK(x,y) n(x,s) n(y,s)\,dx\,dy\,ds \\
&= \int_{\mathbb{R}_+} \varphi(x) n_0(x)\,dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} [\varphi(x+y) - \varphi(x) - \varphi(y)] \\
&\qquad\qquad \times K(x,y) n(x,s) n(y,s)\,dx\,dy\,ds.
\end{aligned}
\tag{2.13}
$$

Notice that all the integrals above are convergent, since for example our test function φ satisfies that $|\varphi(x+y) - \varphi(x) - \varphi(y)| \le Axy$ for some constant A. This completes the proof. $\square$

Equation (MS) can be interpreted as the evolution equation of the time marginals of a pure jump Markov process. In order to exploit this remark, we will associate to (MS) a martingale problem. We begin with some notation.

NOTATION 2.4. Let $T_0 \le \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. Denote by $\mathbb{D}^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$ the set of positive nondecreasing càdlàg functions from $[0,T_0)$ into $\mathcal{H}_{Q_0}$. We denote by $\mathcal{P}_1^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$ the set of probability measures Q on $\mathbb{D}^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$ such that

$$
Q\big(\{x \in \mathbb{D}^{\uparrow}([0,T_0), \mathcal{H}_{Q_0}); x(0) > 0\}\big) = 1 \tag{2.14}
$$

and, for all $t < T_0$,

$$
\int_{x\in\mathbb{D}^{\uparrow}([0,T_0),\mathcal{H}_{Q_0})} x(t)\,Q(dx) = \int_{x\in\mathbb{D}^{\uparrow}([0,T_0),\mathcal{H}_{Q_0})} \Big(\sup_{s\in[0,t]} x(s)\Big) Q(dx) < \infty. \tag{2.15}
$$

The last equality comes naturally from the fact that $x(t)$ is nondecreasing.

DEFINITION 2.5. Let $T_0 \le \infty$, and $Q_0 \in \mathcal{P}_1$ be fixed. Consider $Q \in \mathcal{P}_1^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$, and denote by Q_s its time marginal at s. Let Z be the canonical process of $\mathbb{D}^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$. We say that Q is a solution to the martingale problem

(MP) on $[0, T_0)$ if for all $t \in [0, T_0)$, $\sup_{s\in[0,t]} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K(x, y)Q_s(dx)Q_s(dy) < \infty$, and for all $\varphi \in C_b^1(\mathbb{R}_+)$ the process,

$$\varphi(Z_t) - \varphi(Z_0) - \int_0^t \int_{\mathbb{R}_+} [\varphi(Z_s + y) - \varphi(Z_s)] \frac{K(Z_s, y)}{y} Q_s(dy)\, ds, \tag{2.16}$$

defined for $t \in [0, T_0)$, is a Q-L^1-martingale.

By taking expectations in (2.16), we obtain, using the fact that the expectation of a martingale starting from 0 is 0, the following remark:

REMARK 2.6. Let Q be a solution to the martingale problem (MP) on $[0, T_0)$. For $t \in [0, T_0)$, let Q_t be its time marginal. Then $(Q_t)_{t\in[0,T_0)}$ is a weak solution of (MS) with initial condition Q_0.

We are now seeking for a pathwise representation of the martingale problem (MP). To this aim, let us introduce some more notation. The main ideas of the following notation and definitions come from Tanaka [19].

NOTATION 2.7.

1. We consider two probability spaces: $(\Omega, \mathcal{F}, \mathbb{P})$ is an abstract space and $([0, 1], \mathcal{B}[0, 1], d\alpha)$ is an auxiliary space (here, $d\alpha$ denotes the Lebesgue measure). In order to avoid confusion, the expectation on $[0, 1]$ will be denoted $\mathbb{E}_\alpha$, the laws $\mathcal{L}_\alpha$, the processes will be called α-processes, etc.
2. Let $T_0 \le \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. A nondecreasing positive càdlàg adapted process $(X_t(\omega))_{t\in[0,T_0)}$ is said to belong to $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$ if its law belongs to $\mathcal{P}_1^{\uparrow}([0, T_0), \mathcal{H}_{Q_0})$.

   In the same way, a nondecreasing positive càdlàg α-process $(\tilde{X}_t(\alpha))_{t\in[0,T_0)}$ is said to belong to $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$-$\alpha$ if its α-law belongs to $\mathcal{P}_1^{\uparrow}([0, T_0), \mathcal{H}_{Q_0})$.

DEFINITION 2.8. Let $T_0 \le \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. We say that $(X_0, X, \tilde{X}, N)$ is a solution to the problem (SDE) on $[0, T_0)$ if:

(a) $X_0 : \Omega \to \mathbb{R}_+$ is a random variable whose law is Q_0;

(b) $X_t(\omega) : [0, T_0) \times \Omega \to \mathbb{R}_+$ is a $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$-process;

(c) $\tilde{X}_t(\alpha) : [0, T_0) \times [0, 1] \to \mathbb{R}_+$ is a $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$-$\alpha$-process;

(d) $N(\omega, dt, d\alpha, dz)$ is a Poisson measure on $[0, T_0) \times [0, 1] \times \mathbb{R}_+$ with intensity measure $dt\, d\alpha\, dz$ and is independent of X_0;

(e) X and $\tilde{X}$ have the same law on their respective probability spaces: $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ (this equality holds in $\mathcal{P}_1^{\uparrow}([0, T_0), \mathcal{H}_{Q_0}))$;

(f) for all $t \in [0, T_0)$, $\sup_{s\in[0,t]} \mathbb{E}\mathbb{E}_\alpha(K(X_s, \tilde{X}_s)) < \infty$;

(g) finally, the following SDE is satisfied on $[0, T_0)$:

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \le \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, dz). \tag{2.17}$$

The motivation of this definition is the following:

PROPOSITION 2.9. *Let* $(X_0, X, \tilde{X}, N)$ *be a solution to* (SDE) *on* $[0, T_0)$. *Then the law* $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ *satisfies the martingale problem* (MP) *on* $[0, T_0)$ *with initial condition* $Q_0 = \mathcal{L}(X_0)$. *Hence* $\{\mathcal{L}(X_t)\}_{t\in[0,T_0)}$ *is a solution to the modified Smoluchowski equation* (MS) *with initial condition* Q_0.

Before proving rigorously this result, we explain its main intuition: why is it natural to choose $\{X_t\}_{t\ge 0}$ satisfying (SDE), in order to obtain a stochastic process whose law is solution to the modified Smoluchowski equation (MS)?

We wish the law Q_t of X_t to describe the evolution of the distribution of particles's masses in the system. A natural way to do this is to choose one particle randomly, and to use a random (but natural) coagulation dynamic. Thus, X_t should be understood as the evolution of the size of a sort of "typical" particle. Of course, X_0 must follow the initial distribution Q_0. Afterwards, at some random instants, which are typically Poissonian instants (for Markovian reasons), coalescence phenomena occur. Let τ be one of these instants. At this instant, we choose another particle, randomly in the system, and we denote by $\tilde{X}_\tau(\alpha)$ its size. Then we describe the coagulation as $X_\tau = X_{\tau-} + \tilde{X}_\tau(\alpha)$. The indicator function in (2.17) allows to control the frequency of the coagulations.

Thus, from a time-evolution point of view, $X_\bullet$ mimics randomly the evolution of the size of one particle, its law is given by the (deterministic) "true" distribution of the sizes in the system at time t, which is exactly the solution of (MS).

PROOF OF PROPOSITION 2.9. Let φ be a $C_b^1(\mathbb{R}_+)$ function. Then for all $t \in [0, T_0)$,

$$\begin{aligned}
\varphi(X_t) &= \varphi(X_0) + \sum_{s\le t} [\varphi(X_s) - \varphi(X_{s-})] \\
&= \varphi(X_0) + \int_0^t \int_0^1 \int_0^\infty \Bigg[\varphi\Bigg(X_{s-} + \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \le \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}}\Bigg) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \varphi(X_{s-})\Bigg] N(ds, d\alpha, dz) \\
&= \varphi(X_0) + \int_0^t \int_0^1 \int_0^\infty [\varphi(X_{s-} + \tilde{X}_{s-}(\alpha)) - \varphi(X_{s-})] \\
&\qquad\qquad\qquad \times \mathbb{1}_{\left\{z \le \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, dz).
\end{aligned}$$

Hence

$$M_t^\varphi = \varphi(X_t) - \varphi(X_0) \\ - \int_0^t \int_0^1 \int_0^\infty [\varphi(X_s + \tilde{X}_s(\alpha)) - \varphi(X_s)] \mathbb{1}_{\left\{z \le \frac{K(X_s, \tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right\}} \, dz \, d\alpha \, ds \tag{2.18}$$

can be written as a stochastic integral with respect to the compensated Poisson measure, and thus is a martingale. But

$$\begin{aligned} M_t^\varphi &= \varphi(X_t) - \varphi(X_0) \\ &\quad - \int_0^t \mathbb{E}_\alpha\left[(\varphi(X_s + \tilde{X}_s(\alpha)) - \varphi(X_s)) \frac{K(X_s, \tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right] ds \\ &= \varphi(X_t) - \varphi(X_0) - \int_0^t \int_{\mathbb{R}_+} [\varphi(X_s + y) - \varphi(y)] \frac{K(X_s, y)}{y} Q_s(dy) \, ds \end{aligned} \tag{2.19}$$

where $Q_s = \mathcal{L}_\alpha(\tilde{X}_s) = \mathcal{L}(X_s)$. We have proved that $\mathcal{L}(X)$ satisfies (MP) on $[0, T_0)$. $\square$

Let us now state a hypothesis which will allow to prove existence results for (SDE).

(H_β): The initial condition Q_0 belongs to $\mathcal{P}_1$. The symmetric kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is locally Lipschitz continuous on $(\mathcal{H}_{Q_0})^2$, and satisfies, for some constant $C_K < \infty$ and some $\beta \in [0, 1]$,

$$K(x, y) \le C_K(1 + x + y + x^\beta y^\beta). \tag{2.20}$$

Two different situations will appear according to $\beta = 1/2$ or $\beta = 1$. We will always prove the results for the case $\beta = 1$ the other one being similar and easier to treat. Let us also remark that all results for $\beta = 1/2$ apply also for $0 \le \beta \le 1/2$ and similarly the ones for $\beta = 1$ are true for $1/2 < \beta \le 1$.

Notice also that in the discrete case, $\mathcal{H}_{Q_0}$ is contained in $\mathbb{N}^*$, so that we don't need the local Lipschitz continuity condition.

3. Existence results for (SDE). The aim of this section is to prove the following result.

THEOREM 3.1. *Let Q_0 satisfy $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$. Assume* ($\mathrm{H}_\beta$).

(i) *If $\beta = 1/2$ then there exists a solution $(X_0, X, \tilde{X}, N)$ to* (SDE), *on* $[0, T_0)$, *where $T_0 = \infty$.*

(ii) *If $\beta = 1$ then there exists a solution $(X_0, X, \tilde{X}, N)$ to* (SDE), *on* $[0, T_0)$, *where $T_0 = 1/C_K(1 + \mathbb{E}(X_0))$.*

REMARK 3.2. Assume (H_β). From now on for $\beta = 1/2$ let $T_0 = \infty$ and for $\beta = 1$ let $T_0 = 1/C_K(1 + \mathbb{E}(X_0))$.

We obtain the following corollary, which states a new existence result for the continuous Smoluchowski equation, enabling some initial conditions $n_0(x)$ which are not integrable at $x = 0$. We express this in terms of measures; see Norris [16].

COROLLARY 3.3. *Consider a nonnegative measure μ_0 on $\mathbb{R}_+^*$ satisfying that $\int_{\mathbb{R}_+} x\mu_0(dx) = 1$, $\int_{\mathbb{R}_+} x^3\mu_0(dx) < \infty$ and consider the associated probability measure $Q_0(dx) = x\mu_0(dx)$. Assume* (H_β), *and consider the associated T_0* (*see Remark* 3.2).

Then there exists a weak solution $\{\mu_t\}_{t\geq 0}$ to the Smoluchowski equation, in the sense that:

(i) *for all $t < T_0$, $\int_{\mathbb{R}_+} x\mu_t(x)\,dx = 1$ and $\sup_{s\in[0,t]}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} xyK(x,y) \times \mu_s(dx)\mu_s(dy) < \infty$,*

(ii) *for all test function φ on $\mathbb{R}_+$ such that $\varphi(x)/x$ belongs to $C_b^1(\mathbb{R}_+)$,*

$$\begin{aligned}
&\int_{\mathbb{R}_+} \varphi(x)\mu_t(dx)\\
&\quad= \int_{\mathbb{R}_+} \varphi(x)\mu_0(dx)\\
&\qquad+ \frac{1}{2}\int_0^t \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} [\varphi(x+y) - \varphi(x) - \varphi(y)]K(x,y)\mu_s(dx)\mu_s(dy)\,ds.
\end{aligned} \tag{3.1}$$

The proof is straightforward: using Theorem 3.1, Proposition 2.9 and Remark 2.6, we obtain the existence of a solution $\{Q_t\}_{t\in[0,T_0)}$ to (MS), which can be rewritten in terms of $\mu_t(dx) = Q_t(dx)/x$ exactly as in the corollary.

This result is new since we do not need to suppose that $\int_{\mathbb{R}_+} \mu_0(dx)$ is finite.

From Theorem 3.1 we see that for $\beta = 1/2$ we obtain an existence result on $[0, \infty)$. This is not the case if $\beta = 1$. Indeed, it is classical that for $\beta = 1$ there is gelation in finite time. More precisely, Jeon [12] proved for the discrete case that if $K(i, j) \geq i^\beta j^\beta$ for some $1/2 < \beta < 1$, if we denote by $n(k,t)$ a solution to (SD), we have that the gelation time T_{gel} defined by

$$T_{\mathrm{gel}} = \inf\left\{t \geq 0;\ \sum_{k\geq 1} k^2 n(k,t) = \infty\right\} \tag{3.2}$$

is finite. With our notation this becomes

$$T_{\mathrm{gel}} = \inf\{t \geq 0;\ \mathbb{E}(X_t) = \infty\} < \infty. \tag{3.3}$$

It is thus clear that an existence result on $[0, \infty)$ cannot be proved under the assumption (H_β) for $\beta = 1$.

Finally, notice that for $\beta = 1$, $T_0 = 1/C_K(1 + \mathbb{E}(X_0))$ is not the exact gelation time, except if $K(x, y) = C_K(1 + x + y + xy)$: since we only assume an upper bound on K, we are only able to prove an existence result for (SDE) on $[0, T_0)$, for some $T_0 \leq T_{\mathrm{gel}}$. We however will give exact gelation times corresponding to a class of coagulation kernels for which explicit computations are easy. In such cases, our existence result will easily extend to $[0, T_{\mathrm{gel}})$.

Because the coefficients of (SDE) are not globally Lipschitz continuous, Theorem 3.1 is not easy to prove. Due to the nonlinearity, a direct construction is difficult. Thus, in a first proposition, we prove a result, which combined with Proposition 2.9 shows that the existence (respectively uniqueness in law) for (SDE) is equivalent to existence (respectively uniqueness) for (MP). It will thus be sufficient to prove an existence result for (MP).

Next, we use a cutoff procedure, which renders the coefficients of our equation globally Lipschitz continuous: we obtain the existence of a solution X^ε to a cutoff equation $(\mathrm{SDE})_\varepsilon$. Tightness and uniform integrability results allow to prove that the family $\mathcal{L}(X^\varepsilon)$ has limiting points, and that these limit points satisfy (MP).

As said previously, we begin with a proposition, which, combined with Proposition 2.9, shows a sort of equivalence between (MP) and (SDE).

PROPOSITION 3.4. *Let Q_0 belong to $\mathcal{P}_1$. Assume that $Q \in \mathcal{P}_1^{\uparrow}([0, T_0), \mathcal{H}_{Q_0})$ is a solution to* (MP) *on $[0, T_0)$ with initial condition Q_0, for some $T_0 \leq \infty$.*

Consider any $L_1^{T_0,\uparrow}(\mathcal{H}_{Q_0})$-$\alpha$-process $\tilde{X}$ such that $\mathcal{L}_\alpha(\tilde{X}) = Q$. Consider also the canonical process Z of $\mathbb{D}^{\uparrow}([0, T_0), \mathcal{H}_{Q_0})$. Then there exists, on an enlarged probability space (*from the canonical one*), *a Poisson measure $N(\omega, dt, d\alpha, dz)$, independent of Z_0* (*all of this under Q*), *such that $(Z_0, Z, \tilde{X}, N)$ is a solution to* (SDE) (*still under Q*).

This kind of result is now standard and relies on representation Theorems for point processes, we refer to Desvillettes, Graham and Méléard [5] or to the original paper of Tanaka [20].

In order to prove Theorem 3.1, we first consider a simpler problem with cutoff.

For Q_0 in $\mathcal{P}_1$, we define a solution $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ to $(\mathrm{SDE})_\varepsilon$ exactly in the same way as in Definition 2.8, but replacing (2.17) by

$$
\begin{aligned}
X_t^\varepsilon = X_0 + \int_0^t \int_0^1 \int_0^\infty & \left(\tilde{X}_{s-}^\varepsilon(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} \right) \\
& \times \mathbb{1}_{\left\{ z \leq \frac{K(X_{s-}^\varepsilon \wedge (1/\varepsilon), \tilde{X}_{s-}^\varepsilon(\alpha) \wedge (1/\varepsilon))}{\tilde{X}_{s-}^\varepsilon(\alpha) \vee \varepsilon \wedge (1/\varepsilon)} \right\}} N(ds, d\alpha, dz)
\end{aligned} \tag{3.4}
$$

under the conditions $\mathcal{L}(X^\varepsilon) \in \mathcal{P}_1^{\uparrow}([0, T_0), \mathcal{H}_{Q_0})$ and $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = \mathcal{L}(X^\varepsilon)$.

We begin with an important remark.

REMARK 3.5. We need that for each $\varepsilon > 0$ and for $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ a solution to $(\mathrm{SDE})_\varepsilon$, X^ε takes its values in $\mathcal{H}_{Q_0}$. Indeed, the regularity assumption (H_β)

on K holds only on $\mathcal{H}_{Q_0}$. Hence, in (3.4), $x \vee \varepsilon \wedge (1/\varepsilon)$ is only a notation, and its rigorous definition is, for any $x \in \mathcal{H}_{Q_0}$ and any $\varepsilon > 0$,

$$x \vee \varepsilon \wedge (1/\varepsilon) = \begin{cases} \inf\{y \in \mathcal{H}_{Q_0};\, y \geq \varepsilon\}, & \text{if } 0 \leq x \leq \varepsilon, \\ x, & \text{if } x \in [\varepsilon, 1/\varepsilon], \\ \sup\{y \in \mathcal{H}_{Q_0};\, y \leq 1/\varepsilon\}, & \text{if } 1/\varepsilon \leq x. \end{cases} \tag{3.5}$$

Of course, $x \wedge (1/\varepsilon)$ is defined in the same way. With these definitions, $x \vee \varepsilon \wedge (1/\varepsilon)$ and $x \wedge (1/\varepsilon)$ belong to $\mathcal{H}_{Q_0}$ for any $x \in \mathcal{H}_{Q_0}$, $\varepsilon > 0$.

We now prove an existence result for $(\mathrm{SDE})_\varepsilon$.

PROPOSITION 3.6. *Let $Q_0 \in \mathcal{P}_1$ and $\varepsilon > 0$. Assume* (H_β). *Let X_0 be a random variable whose law is Q_0 and N be a Poisson measure independent of X_0. Then there exists a solution $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ to* $(\mathrm{SDE})_\varepsilon$ *on* $[0, \infty)$.

PROOF. The proof mimics that of Tanaka, who proved in [19] a similar result in the case of a nonlinear SDE related to the Boltzmann equation. We refer to the more recent work of Desvillettes, Graham and Méléard [5] for a detailed proof in a simpler one-dimensional case. We thus only point the main ideas of the proof.

We introduce the following nonclassical Picard approximations. First, we consider the process $X^{0,\varepsilon} \equiv X_0$, and any α-process $\tilde{X}^{0,\varepsilon}$ such that $\mathcal{L}_\alpha(\tilde{X}^{0,\varepsilon}) = \mathcal{L}(X^{0,\varepsilon})$.

Once everything is built up to n, we set

$$\begin{aligned} X_t^{n+1,\varepsilon} = X_0 + \int_0^t \int_0^1 \int_0^\infty & \left(\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} \right) \\ & \times \mathbb{1}_{\left\{ z \leq \frac{K(X_{s-}^{n,\varepsilon} \wedge (1/\varepsilon), \tilde{X}_{s-}^{n,\varepsilon}(\alpha) \wedge (1/\varepsilon))}{\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge (1/\varepsilon)} \right\}} N(ds, d\alpha, dz) \end{aligned} \tag{3.6}$$

and we consider any α-process $\tilde{X}^{n+1,\varepsilon}$ such that

$$\mathcal{L}_\alpha(\tilde{X}^{n+1,\varepsilon} \mid \tilde{X}^{0,\varepsilon}, \dots, \tilde{X}^{n,\varepsilon}) = \mathcal{L}(X^{n+1,\varepsilon} \mid X^{0,\varepsilon}, \dots, X^{n,\varepsilon}). \tag{3.7}$$

One easily checks recursively that for each n, $X^{n,\varepsilon}$ is an $L_1^{\infty,\uparrow}(\mathcal{H}_{Q_0})$-process.

Let us show now that the sequence $\{X^{n,\varepsilon}\}_n$ is Cauchy in $L_1^{\infty,\uparrow}(\mathcal{H}_{Q_0})$. We set $\varphi_n(t) := \mathbb{E}[\sup_{s\in[0,t]} |X_s^{n+1,\varepsilon} - X_s^{n,\varepsilon}|]$. A simple computation, using the fact that the map

$$(x, y) \mapsto \frac{K(x \wedge (1/\varepsilon), y \wedge (1/\varepsilon))}{x \vee \varepsilon \wedge (1/\varepsilon)} \tag{3.8}$$

is globally Lipschitz continuous on $(\mathcal{H}_{Q_0})^2$ [thanks to (H_β)] and the fact that $\int_0^1 |\tilde{X}^{n,\varepsilon}(\alpha) - \tilde{X}^{n-1,\varepsilon}(\alpha)|\, d\alpha \leq \varphi_{n-1}(s)$, gives the existence of a constant A, depending only on ε, such that

$$\varphi_n(t) \leq A \int_0^t \varphi_{n-1}(s)\, ds. \tag{3.9}$$

We conclude, thanks to the usual Gronwall Lemma, that there exists an $L_1^{\infty,\uparrow}(\mathcal{H}_{Q_0})$-process X^ε such that, for any $T < \infty$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^{n,\varepsilon} - X_t^\varepsilon|\Big] \underset{n\to\infty}{\longrightarrow} 0. \tag{3.10}$$

By construction, the α-law of the sequence of processes $\tilde{X}^{0,\varepsilon}, \dots, \tilde{X}^{n,\varepsilon}, \dots$ is the same as the law of the sequence $X^{0,\varepsilon}, \dots, X^{n,\varepsilon}, \dots$. We thus deduce the existence of an $L_1^{\infty,\uparrow}(\mathcal{H}_{Q_0})$-$\alpha$-process $\tilde{X}^\varepsilon$ such that $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = \mathcal{L}(X^\varepsilon)$, and such that for all $T < \infty$,

$$\mathbb{E}_\alpha\Big[\sup_{t\in[0,T]} |\tilde{X}_t^{n,\varepsilon} - \tilde{X}_t^\varepsilon|\Big] \underset{n\to\infty}{\longrightarrow} 0. \tag{3.11}$$

Letting n go to infinity in (3.6) concludes the proof. $\square$

We now prove the tightness of the family $\{\mathcal{L}(X^\varepsilon)\}_\varepsilon$.

LEMMA 3.7. *Assume* (H_β). *For* $\beta = 1/2$ *or* $\beta = 1$ *consider the corresponding* T_0. *Consider a family* $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ *of solutions to* $(\mathrm{SDE})_\varepsilon$. *Then, for all* $T < T_0$,

$$\sup_{\varepsilon>0} \mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^\varepsilon|\Big] = \sup_{\varepsilon>0} \mathbb{E}_\alpha\Big[\sup_{t\in[0,T]} |\tilde{X}_t^\varepsilon|\Big] < \infty. \tag{3.12}$$

Furthermore, the family $\mathcal{L}(X^\varepsilon) = \mathcal{L}_\alpha(\tilde{X}^\varepsilon)$ *of probability measures on* $\mathbb{D}^\uparrow([0, T_0), \mathcal{H}_{Q_0})$ *is tight, and any limiting point* Q *of a convergent subsequence is the law of a quasi-left continuous process* (*for the definition see Jacod and Shiryaev* [11], *page* 22).

PROOF. Let us prove the result under (H_β) for $\beta = 1$, the case $\beta = 1/2$ being similar. We first check (3.12). Setting

$$f_\varepsilon(t) = \mathbb{E}\Big[\sup_{s\in[0,t]} |X_s^\varepsilon|\Big], \tag{3.13}$$

it is immediate, since the processes are positive and nondecreasing and since for each ε, $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = \mathcal{L}(X^\varepsilon)$, that

$$f_\varepsilon(t) = \mathbb{E}[X_t^\varepsilon] = \mathbb{E}_\alpha[\tilde{X}_t^\varepsilon]. \tag{3.14}$$

A simple computation, using (3.4), yields that

$$f_\varepsilon(t) = \mathbb{E}(X_0) + \int_0^t \mathbb{E}\mathbb{E}_\alpha\Big[K\Big(X_s^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_s^\varepsilon \wedge \frac{1}{\varepsilon}\Big)\Big]\, ds. \tag{3.15}$$

Under (H_β) with $\beta = 1$, it is clear that

$$\mathbb{E}\mathbb{E}_\alpha\left[K\left(X_s^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_s^\varepsilon \wedge \frac{1}{\varepsilon}\right)\right] \\ \leq C_K\big(1 + 2f_\varepsilon(s) + f_\varepsilon^2(s)\big) = C_K\big(1 + f_\varepsilon(s)\big)^2. \tag{3.16}$$

Lemma A.3 in the Appendix, applied to the function $g_\varepsilon = 1 + f_\varepsilon$, which is clearly continuous [thanks to (3.15)], allows to conclude that for any $t < T_0 = 1/C_K(1 + \mathbb{E}(X_0))$,

$$f_\varepsilon(t) \leq \frac{1 + \mathbb{E}(X_0)}{1 - t/T_0} - 1 \tag{3.17}$$

from which (3.12) is straightforward.

In order to obtain the tightness of the family $\{\mathcal{L}(X^\varepsilon)\}_\varepsilon$, we use the Aldous criterion, which is recalled in the Appendix (Theorem A.1).

We just have to check that for all $T < T_0$ fixed, there exists a constant A_T such that for all $\delta > 0$, all couple of stopping times S and S' satisfying a.s. $0 \leq S \leq S' \leq (S + \delta) \wedge T$, and all ε,

$$\mathbb{E}\big|X_{S'}^\varepsilon - X_S^\varepsilon\big| \leq A_T\delta, \tag{3.18}$$

the constant A_T being independent of ε, δ, S and S'. This is not hard. Indeed,

$$\big|X_{S'}^\varepsilon - X_S^\varepsilon\big| \\ = \int_{(S,S']}\left(\tilde{X}_{u-}^\varepsilon(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}\right)\mathbb{1}_{\left\{z \leq \frac{K(X_{u-}^\varepsilon \wedge (1/\varepsilon), \tilde{X}_{u-}^\varepsilon(\alpha) \wedge (1/\varepsilon))}{\tilde{X}_{u-}^\varepsilon(\alpha) \vee \varepsilon \wedge (1/\varepsilon)}\right\}} N(du, d\alpha, dz). \tag{3.19}$$

Since $\mathbb{1}_{(S,S']}(u)$ is predictable (it is left continuous and adapted), we get

$$\mathbb{E}\big[\big|X_{S'}^\varepsilon - X_S^\varepsilon\big|\big] = \mathbb{E}\mathbb{E}_\alpha\left[\int_S^{S'} K\big(X_u^\varepsilon, \tilde{X}_u^\varepsilon(\alpha)\big)\,du\right] \\ \leq \delta \sup_{u \in [0,T]} \mathbb{E}\mathbb{E}_\alpha\big[K(X_u^\varepsilon, \tilde{X}_u^\varepsilon)\big]. \tag{3.20}$$

But thanks to (H_β) for $\beta = 1$ and to (3.12) (since $T < T_0$),

$$\sup_{u \in [0,T]} \mathbb{E}\mathbb{E}_\alpha\big[K(X_u^\varepsilon, \tilde{X}_u^\varepsilon)\big] \leq C_K \sup_{u \in [0,T]} \mathbb{E}\mathbb{E}_\alpha\big[1 + X_u^\varepsilon + \tilde{X}_u^\varepsilon + X_u^\varepsilon\tilde{X}_u^\varepsilon\big] \\ \leq C_K\big[1 + 2\mathbb{E}[X_T^\varepsilon] + \mathbb{E}[X_T^\varepsilon]^2\big] \leq A_T, \tag{3.21}$$

which concludes the proof. $\square$

To prove that any limiting point Q of $\mathcal{L}(X^\varepsilon)$ satisfies (MP), we need also a property of uniform integrability, which will be obtained in the next lemma.

LEMMA 3.8. *Assume that* $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$. *Assume* (H_β), *and following the value of* β *consider the associated* T_0. *Consider a family* $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ *of solutions to* $(\mathrm{SDE})_\varepsilon$. *Then for all* $T < T_0$ *fixed*,

$$\sup_{\varepsilon>0} \mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^\varepsilon|^2\Big] < \infty. \tag{3.22}$$

PROOF. For $k \in \mathbb{N}^*$, we define

$$g_k^\varepsilon(t) = \mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^\varepsilon|^k\Big] = \mathbb{E}\big[(X_T^\varepsilon)^k\big]. \tag{3.23}$$

For all $t < T_0$,

$$\begin{aligned}(X_t^\varepsilon)^2 &= X_0^2 + \sum_{s\le t}\Big(\big(X_{s-}^\varepsilon + \Delta X_s^\varepsilon\big)^2 - \big(X_{s-}^\varepsilon\big)^2\Big)\\ &= X_0^2 + \int_0^t\int_0^1\int_0^\infty \bigg(2X_{s-}^\varepsilon\bigg(\tilde{X}_{s-}^\varepsilon(\alpha)\vee\varepsilon\wedge\frac{1}{\varepsilon}\bigg) + \bigg(\tilde{X}_{s-}^\varepsilon(\alpha)\vee\varepsilon\wedge\frac{1}{\varepsilon}\bigg)^2\bigg)\\ &\qquad\times \mathbb{1}_{\left\{z\le \frac{K(X_{s-}^\varepsilon\wedge(1/\varepsilon),\tilde{X}_{s-}^\varepsilon(\alpha)\wedge(1/\varepsilon))}{\tilde{X}_{s-}^\varepsilon(\alpha)\vee\varepsilon\wedge(1/\varepsilon)}\right\}} N(ds,d\alpha,dz).\end{aligned} \tag{3.24}$$

Hence

$$g_2^\varepsilon(t) = \mathbb{E}(X_0^2) + \int_0^t \mathbb{E}\mathbb{E}_\alpha\bigg[K\bigg(X_s^\varepsilon\wedge\frac{1}{\varepsilon}, \tilde{X}_s^\varepsilon\wedge\frac{1}{\varepsilon}\bigg)\big(2X_s^\varepsilon + (\tilde{X}_s^\varepsilon\vee\varepsilon)\big)\bigg]ds. \tag{3.25}$$

Let us complete the proof for (H_β) with $\beta = 1$, the other case being similar. Let thus $T < T_0$ be fixed. Using the fact that $\mathcal{L}(X^\varepsilon) = \mathcal{L}_\alpha(\tilde{X}^\varepsilon)$ and (3.12), we obtain the existence of a constant A_T, not depending on ε, such that for all $t \le T$,

$$\begin{aligned}g_2^\varepsilon(t) &\le \mathbb{E}(X_0^2) + 3C_K\int_0^t \mathbb{E}\mathbb{E}_\alpha\big[(\tilde{X}_s^\varepsilon+\varepsilon)\big(1+X_s^\varepsilon+\tilde{X}_s^\varepsilon+X_s^\varepsilon\tilde{X}_s^\varepsilon\big)\big]ds\\ &\le \mathbb{E}(X_0^2) + A_T\int_0^t\big[1+g_2^\varepsilon(s)\big]ds.\end{aligned} \tag{3.26}$$

The usual Gronwall Lemma allows us to conclude the proof. $\square$

The following lemma, associated with Proposition 3.4, will conclude the proof of Theorem 3.1.

LEMMA 3.9. *Let* Q_0 *satisfy* $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$. *Assume* (H_β) *and consider the corresponding* T_0. *Consider a family* $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ *of solutions to* $(\mathrm{SDE})_\varepsilon$, *and a limiting point* Q *of the tight family* $\mathcal{L}(X^\varepsilon) = \mathcal{L}_\alpha(\tilde{X}^\varepsilon)$. *Then* Q *is a solution to* (MP) *on* $[0, T_0)$, *with initial condition* $Q_0 = \mathcal{L}(X_0)$.

PROOF. We prove the result for $\beta = 1$. The other case is simpler. Let Q be the limit of a sequence of $Q^k = \mathcal{L}(X^{\varepsilon_k})$, ε_k being a sequence of positive real numbers decreasing to 0.

We have to check that, for any $\phi \in C_b^1(\mathbb{R}_+)$, any $g_1, \dots, g_l \in C_b(\mathbb{R}_+)$ and any $0 \le s_1 \le \cdots \le s_l < s < t < T_0$,

$$\langle Q \otimes Q, F \rangle = 0 \tag{3.27}$$

where F is the map from $\mathbb{D}^{\uparrow}([0, T_0), \mathcal{H}_{Q_0}) \times \mathbb{D}^{\uparrow}([0, T_0), \mathcal{H}_{Q_0})$ defined by

$$\begin{aligned} F(x, y) &= g_1\big(x(s_1)\big) \times \cdots \times g_l\big(x(s_l)\big) \\ &\quad \times \left\{\phi\big(x(t)\big) - \phi\big(x(s)\big) - \int_s^t \big[\phi\big(x(u) + y(u)\big) - \phi\big(x(u)\big)\big] \right. \\ &\qquad\qquad \left. \times \frac{K(x(u), y(u))}{y(u)}\, du\right\}. \end{aligned} \tag{3.28}$$

It is clear from the definition of the process X^{ε_k} that for any k,

$$\langle Q^k \otimes Q^k, F^k \rangle = 0, \tag{3.29}$$

where F^k is defined by

$$\begin{aligned} F^k(x, y) &= g_1\big(x(s_1)\big) \times \cdots \times g_l\big(x(s_l)\big) \\ &\quad \times \left\{\phi\big(x(t)\big) - \phi\big(x(s)\big) - \int_s^t \left[\phi\left(x(u) + y(u) \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}\right) - \phi\big(x(u)\big)\right] \right. \\ &\qquad\qquad \left. \times \frac{K(x(u) \wedge (1/\varepsilon_k), y(u) \wedge (1/\varepsilon_k))}{y(u) \vee \varepsilon_k \wedge (1/\varepsilon_k)}\, du\right\}. \end{aligned} \tag{3.30}$$

It thus suffices to prove that $\langle Q^k \otimes Q^k, F^k \rangle$ tends to $\langle Q \otimes Q, F \rangle$ as k tends to infinity. We split the proof into two steps.

Step 1. Let us first check that,

$$\big\langle Q^k \otimes Q^k, |F - F^k| \big\rangle \underset{k \to \infty}{\longrightarrow} 0. \tag{3.31}$$

By definition,

$$\begin{aligned} &\big\langle Q^k \otimes Q^k, |F - F^k| \big\rangle \\ &\quad = \mathbb{E}\mathbb{E}_\alpha\left[\left| g_1\big(X^{\varepsilon_k}(s_1)\big) \times \cdots \times g_l\big(X^{\varepsilon_k}(s_l)\big) \right.\right. \\ &\qquad \times \int_s^t \left\{\left[\varphi\left(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}\right) - \varphi(X_u^{\varepsilon_k})\right]\right. \\ &\qquad\qquad \times \frac{K(X_u^{\varepsilon_k} \wedge (1/\varepsilon_k), \tilde{X}_u^{\varepsilon_k} \wedge (1/\varepsilon_k))}{\tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge (1/\varepsilon_k)} \\ &\qquad\qquad \left.\left.\left. - \big[\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})\big] \frac{K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}}\right\} du \right|\right]. \end{aligned} \tag{3.32}$$

Hence, for some constant A, $\langle Q^k \otimes Q^k, |F - F^k|\rangle$ is smaller than

$$
\begin{aligned}
& A\mathbb{E}\mathbb{E}_\alpha\biggl[\int_s^t \biggl|\frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge (1/\varepsilon_k) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge (1/\varepsilon_k)} \\
& \qquad\qquad - \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}}\biggr| K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k})\, du\biggr] \\
& \quad + A\mathbb{E}\mathbb{E}_\alpha\biggl[\int_s^t \frac{|\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge (1/\varepsilon_k)) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge (1/\varepsilon_k)} \\
& \qquad\qquad \times \biggl| K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k}) - K\biggl(X_u^{\varepsilon_k} \wedge \frac{1}{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k} \wedge \frac{1}{\varepsilon_k}\biggr)\biggr|\, du\biggr] \\
& \quad = A(I_{\varepsilon_k} + J_{\varepsilon_k}),
\end{aligned}
\tag{3.33}
$$

with obvious notation for I_{ε_k} and J_{ε_k}. As φ' is bounded, we obtain, using (H_β) for $\beta = 1$,

$$
\begin{aligned}
J_{\varepsilon_k} &\le 2\|\varphi'\|_\infty \mathbb{E}\mathbb{E}_\alpha\biggl[\int_s^t \bigl(\mathbb{1}_{\{X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}} + \mathbb{1}_{\{\tilde{X}_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}}\bigr) \\
& \qquad\qquad \times (1 + X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} + X_u^{\varepsilon_k}\tilde{X}_u^{\varepsilon_k})\, du\biggr] \\
&\le A\Bigl[\mathbb{P}(X_t^{\varepsilon_k} > 1/\varepsilon_k) + \mathbb{E}\bigl[X_t^{\varepsilon_k}\mathbb{1}_{\{X_t^{\varepsilon_k} > 1/\varepsilon_k\}}\bigr]\Bigr].
\end{aligned}
\tag{3.34}
$$

The uniform integrability obtained in Lemma 3.8 allows to conclude that J_{ε_k} tends to 0.

Let us now bound I_{ε_k} from above. Remark first that

$$
I_{\varepsilon_k} \le I_{\varepsilon_k}^1 + I_{\varepsilon_k}^2
\tag{3.35}
$$

where

$$
\begin{aligned}
I_{\varepsilon_k}^1 = A\mathbb{E}\mathbb{E}_\alpha\biggl[\int_s^t \mathbb{1}_{\{\tilde{X}_u^{\varepsilon_k} < \varepsilon_k\}} K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k}) \biggl| & \frac{\varphi(X_u^{\varepsilon_k} + \varepsilon_k) - \varphi(X_u^{\varepsilon_k})}{\varepsilon_k} \\
& - \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}}\biggr|\, du\biggr]
\end{aligned}
\tag{3.36}
$$

and

$$
\begin{aligned}
I_{\varepsilon_k}^2 = A\mathbb{E}\mathbb{E}_\alpha\biggl[\int_s^t \mathbb{1}_{\{\tilde{X}_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}} K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k}) \biggl| & \frac{\varphi(X_u^{\varepsilon_k} + (1/\varepsilon_k)) - \varphi(X_u^{\varepsilon_k})}{1/\varepsilon_k} \\
& - \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}}\biggr|\, du\biggr].
\end{aligned}
\tag{3.37}
$$

The second term is similar to J_{ε_k}, and thus goes to 0 as k tends to infinity. Using (H_β) with $\beta = 1$ and (3.12), we see that the first term is smaller than

$$
\begin{aligned}
I^1_{\varepsilon_k} &\le 2A\|\varphi'\|_\infty \int_s^t \mathbb{E}\mathbb{E}_\alpha\big[\mathbb{1}_{\{\tilde{X}^{\varepsilon_k}_u < \varepsilon_k\}}(1 + X^{\varepsilon_k}_u + \tilde{X}^{\varepsilon_k}_u + \tilde{X}^{\varepsilon_k}_u X^{\varepsilon_k}_u)\big]\,du \\
&\le A\int_0^t \mathbb{P}(X^{\varepsilon_k}_u < \varepsilon_k)\,du \le At\mathbb{P}(X_0 < \varepsilon_k),
\end{aligned} \tag{3.38}
$$

where the last inequality comes from the fact that the process X^{ε_k} is nondecreasing. This goes to 0, because $X_0 > 0$ a.s. Step 1 is complete.

Step 2. It remains to prove that

$$
\langle Q^k \otimes Q^k, F\rangle \underset{k\to\infty}{\longrightarrow} \langle Q \otimes Q, F\rangle. \tag{3.39}
$$

This convergence would be obvious if F was continuous and bounded on $\mathbb{D}^\uparrow([0,T_0),\mathcal{H}_{Q_0}) \times \mathbb{D}^\uparrow([0,T_0),\mathcal{H}_{Q_0})$, thanks to the definition of the convergence in law. The map F is not continuous on $\mathbb{D}^\uparrow([0,T_0),\mathcal{H}_{Q_0}) \times \mathbb{D}^\uparrow([0,T_0),\mathcal{H}_{Q_0})$, but only on $\mathcal{C} \times \mathcal{C}$, where

$$
\begin{aligned}
\mathcal{C} = \big\{x \in \mathbb{D}^\uparrow([0,T_0),\mathcal{H}_{Q_0});\\
\Delta x(s_1) = \cdots = \Delta x(s_l) = \Delta x(s) = \Delta x(t) = 0\big\}.
\end{aligned} \tag{3.40}
$$

Thanks to Lemma 3.7, Q is the law of a quasi-left continuous process, thus $Q(\mathcal{C}) = 1$, and hence F is $Q \otimes Q$-a.e. continuous. This implies that for any positive constant A,

$$
\langle Q^k \otimes Q^k, F \wedge A \vee (-A)\rangle \underset{k\to\infty}{\longrightarrow} \langle Q \otimes Q, F \wedge A \vee (-A)\rangle \tag{3.41}
$$

because $F \wedge A \vee (-A)$ is $Q \otimes Q$-a.e. continuous and bounded. Thus (3.39) will hold if we prove that

$$
\sup_k \big\langle Q^k \otimes Q^k, |F|\mathbb{1}_{|F|\ge A}\big\rangle \underset{A\to\infty}{\longrightarrow} 0. \tag{3.42}
$$

One can check, after many but easy computations, that

$$
\big\langle Q^k \otimes Q^k, |F|\mathbb{1}_{|F|\ge A}\big\rangle \le B\mathbb{E}\big[X^\varepsilon_t \mathbb{1}_{\{X^\varepsilon_t > \zeta(A)\}}\big] \tag{3.43}
$$

for some constant B and some function $\zeta(A)$ tending to infinity with A. The uniform integrability obtained in Lemma 3.8 allows to conclude that (3.42) holds. Hence (3.39) is valid. This concludes the proof of Step 2 and the proof of the lemma. □

Let us finally conclude the proof of the main result of this section.

PROOF OF THEOREM 3.1. Thanks to Lemma 3.6, there exists a solution $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ to $(\mathrm{SDE})_\varepsilon$ for each ε. From Lemma 3.7, the sequence $\{\mathcal{L}(X^\varepsilon)\}$ is tight, and in particular there exists a sequence ε_k decreasing to 0 such that $\{\mathcal{L}(X^{\varepsilon_k})\}$ tends to some Q. Lemma 3.9 shows that Q satisfies (MP). Finally, Proposition 3.4 allows us to build a solution $(X_0, X, \tilde{X}, N)$ of (SDE). □

REMARK 3.10. Let us remark that our construction procedure for proving Proposition 3.6, gives an existence result of (SDE) without assuming that X and $\tilde{X}$ take values in $\mathcal{H}_{Q_0}$. Our construction gives naturally a process X with values in $\mathcal{H}_{Q_0}$. In particular, for an initial condition X_0 valued in $\mathbb{N}^*$, X takes its values also in $\mathbb{N}^*$.

4. Pathwise behavior of (SDE). In this short section, we try to give an idea on the pathwise properties of X_t, for $(X_0, X, \tilde{X}, N)$ a solution to (SDE). We have very few results on this topic, and the study seems to be difficult. However, we hope that new results will be properly formulated in future works.

We first present an idea about the frequency of the jumps of X_t. How often does a particle in the system coagulate?

The following result, which says that the number of jumps is finite on every compact interval, is not a priori obvious in the continuous case.

PROPOSITION 4.1. *Let Q_0 be such that $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$. Assume* (H$_\beta$) *and consider the corresponding T_0. Let $(X_0, X, \tilde{X}, N)$ be a solution to the corresponding* (SDE). *Assume furthermore that*

$$\int_{\mathbb{R}_+} \frac{1}{x} Q_0(dx) < \infty \tag{4.1}$$

which always holds in the discrete case, and which simply means, in the continuous case, that $\int_{\mathbb{R}_+} n_0(x)\,dx < \infty$.

Denote by $J_t = \sum_{s \le t} \mathbb{1}_{\{\Delta X_s \neq 0\}}$ the number of jumps of X on $[0, t]$. Then for all $t < T_0$, $\mathbb{E}[J_t] < \infty$.

PROOF. Let us again prove the result for $\beta = 1$. Thanks to (2.17), we see that

$$J_t = \int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\left\{z \le \frac{K(X_{s-}, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, dz) \tag{4.2}$$

and hence

$$\mathbb{E}[J_t] = \int_0^t \mathbb{E}\mathbb{E}_\alpha\left[\frac{K(X_s, \tilde{X}_s)}{\tilde{X}_s}\right] ds. \tag{4.3}$$

Using (H$_\beta$) with $\beta = 1$, we obtain

$$\begin{aligned} \mathbb{E}[J_t] &\le C_K \int_0^t \mathbb{E}\mathbb{E}_\alpha\big[1/\tilde{X}_s + X_s/\tilde{X}_s + 1 + X_s\big]\,ds \\ &\le C_K t\big[\mathbb{E}[1/X_0] + \mathbb{E}[X_t]\mathbb{E}[1/X_0] + 1 + \mathbb{E}[X_t]\big] \end{aligned} \tag{4.4}$$

where the last inequality comes from the fact that X is a.s. nondecreasing. This last upper bound is clearly finite, since $t < T_0$ and since we have assumed that $\mathbb{E}(1/X_0) < \infty$. The proof is complete. □

REMARK 4.2. If we do not assume (4.1), we do not know what happens. It however seems that in the (nonexplosive) case where $K(x, y) = 1$ and where $\mathbb{E}(1/X_0) = \infty$, then $X_\bullet$ has infinitely many jumps immediately after 0.

Let us finally talk about the gelation time, defined in (3.3).

This quantity, which can be seen as a L^1-gelation time, has been much studied by the analysts and physicists. It is easily deduced from Theorem 3.1 that under (H_β) with $\beta = 1/2$, $T_{\mathrm{gel}} = \infty$ for any initial condition [satisfying $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$].

In the case $\beta = 1$, under the same assumptions on Q_0, Theorem 3.1 yields that $T_{\mathrm{gel}} \geq T_0 = 1/C_K(1 + \int_{\mathbb{R}_+} x Q_0(dx))$. Of course, we have proved the existence for (SDE) on $[0, T_0)$, because we have only assumed an upper bound for K. But in any particular case where explicit computations could be done, solutions to (SDE) may be constructed on $[0, T_{\mathrm{gel}})$. For example, the following proposition holds.

PROPOSITION 4.3. *Consider $Q_0 \in \mathcal{P}_1$ and suppose that $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$. Assume that $K(x, y) = a + b(x + y) + cxy$, for some nonnegative constants a and b, and for some $c > 0$. Denote by $a_0 = \int_{\mathbb{R}_+} x Q_0(dx)$. Then Theorem* 3.1 *holds by replacing T_0 with T_{gel}, where*:

(i) *if* $\Delta = 4(b^2 - ac) = 0$, *then* $T_{\mathrm{gel}} = \frac{1}{c(a_0+b)}$,

(ii) *if* $\Delta = 4(b^2 - ac) < 0$, *then* $T_{\mathrm{gel}} = \frac{2\pi c}{-\Delta} - \frac{4c}{-\Delta} \arctan(4c\frac{a_0+b}{-\Delta})$,

(iii) *if* $\Delta = 4(b^2 - ac) > 0$, *then* $T_{\mathrm{gel}} = \frac{1}{2\Delta} \ln(\frac{a_0+b/c+\sqrt{\Delta}/c}{a_0+b/c-\sqrt{\Delta}/c})$.

From a probabilistic point of view, the L^1-gelation time is of course important, but we want also to study the stochastic gelation time:

$$\tau_{\mathrm{gel}} = \inf\{t \geq 0; X_t = \infty\}. \tag{4.5}$$

Obviously, $\tau_{\mathrm{gel}} \geq T_{\mathrm{gel}}$ a.s. An interesting question is the following. Under which conditions on Q_0 and K do we have

$$\mathbb{P}(\tau_{\mathrm{gel}} > T_{\mathrm{gel}}) \in (0, 1), \qquad \mathbb{P}(\tau_{\mathrm{gel}} > T_{\mathrm{gel}}) = 0 \quad \text{or} \quad \mathbb{P}(\tau_{\mathrm{gel}} > T_{\mathrm{gel}}) = 1? \tag{4.6}$$

In other words, are there particles of finite (respectively infinite) mass at time T_{gel}? Do all particles have a finite (respectively infinite) mass at time T_{gel}?

We are not able to give a complete answer for the moment. Let us however state and prove the following result.

PROPOSITION 4.4. *Let Q_0 be such that $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$, and let us assume (H_β) with $\beta = 1$. Assume furthermore that $T_{\mathrm{gel}} < \infty$, and that there exists a function $\zeta : \mathrm{Supp}\, Q_0 \mapsto \mathbb{R}_+$ such that, for all $x \in \mathrm{Supp}\, Q_0$*,

$$\sup_{y \in \mathcal{H}_{Q_0}} \frac{K(x, y)}{y} \leq \zeta(x). \tag{4.7}$$

Consider a solution $(X_0, X, \tilde{X}, N)$ *to* (SDE). *Then for any* $t \in [0, \infty)$,

$$\mathbb{P}(\tau_{\mathrm{gel}} > t) > 0. \tag{4.8}$$

This means in particular that there are many particles which have a finite mass at the instant T_{gel}.

Notice that (4.7) is always satisfied in the discrete case, and more generally for any kernel satisfying (H_β) with $\beta = 1$ if $[0, \varepsilon) \cap \operatorname{Supp} Q_0 = \varnothing$ for some $\varepsilon > 0$ (a sort of minimal size).

Notice also that (4.7) is satisfied with any initial condition, if $K(x, y) \leq Cxy$ for some constant $C \in \mathbb{R}_+$.

PROOF OF PROPOSITION 4.4. We will prove a much stronger result: for any $t > 0$, $\mathbb{P}(X_t = X_0) > 0$. To this end, we study the first jump time

$$T_1 = \inf\{s \geq 0; \Delta X_s \neq 0\}. \tag{4.9}$$

By remarking that thanks to (4.7) and (2.17),

$$X_0 \leq X_t \leq X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\{z \leq \zeta(X_{s-})\}} N(ds, d\alpha, dz) \tag{4.10}$$

we deduce that $T_1 \geq S_1$ a.s., where

$$S_1 = \inf\left\{s \geq 0; \int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\{z \leq \zeta(X_0)\}} N(ds, d\alpha, dz) > 0\right\}. \tag{4.11}$$

Since N is a Poisson measure independent of X_0, the random variable

$$\int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\{z \leq \zeta(X_0)\}} N(ds, d\alpha, dz) \tag{4.12}$$

follows, conditionally to X_0, a Poisson distribution of parameter

$$\int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\{z \leq \zeta(X_0)\}} \, ds \, d\alpha \, dz = t\zeta(X_0). \tag{4.13}$$

Hence

$$\mathbb{P}(S_1 > t) = \mathbb{E}\big[\mathbb{P}(S_1 \geq t | X_0)\big] = \mathbb{E}\big[e^{-t\zeta(X_0)}\big] > 0. \tag{4.14}$$

Finally, we conclude that

$$\mathbb{P}(\tau_{\mathrm{gel}} > t) \geq \mathbb{P}(X_t = X_0) = \mathbb{P}(T_1 > t) \geq \mathbb{P}(S_1 > t) > 0 \tag{4.15}$$

which was our aim. $\square$

This concludes the section.

5. About the uniqueness for (SDE). In this section, we deal with the uniqueness in law for (SDE), which is equivalent to the uniqueness for (MP) (see Propositions 2.9 and 3.4). We are not able to prove such uniqueness results by ourselves [except for $K(x, y) = xy$; see the end of this section]. However, we may prove uniqueness by using the results of the analysts. In other words, we may prove uniqueness in law for (SDE) once we know uniqueness for the Smoluchowski equation. We consider first the discrete case.

PROPOSITION 5.1. *Let Q_0 satisfy $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$. Assume* ($\mathrm{H}_\beta$) *and consider the corresponding T_0.*

Assume that $Q_0(\mathbb{N}^*) = 1$ and write Q_0 as $\sum_{k\geq 1} \alpha_k \delta_k(dx)$. Set $n_0(k) = \alpha_k/k$. Assume that uniqueness of a solution to (SC) *with kernel K and initial condition n_0 holds on* $[0, T_0)$. *Then uniqueness of a solution Q to* (MP), *on* $[0, T_0)$ *holds. Hence uniqueness in law holds for* (SDE), *in the sense that any solution $(X_0, X, \tilde{X}, N)$ to* (SDE) *with $\mathcal{L}(X_0) = Q_0$, satisfies $\mathcal{L}(X) = Q$.*

Since we will prove below a similar result in the continuous case, we omit the proof. The following corollary is immediately deduced from Proposition 5.1 and from Heilmann [10].

COROLLARY 5.2. *Assume* ($\mathrm{H}_{1/2}$) *and that $Q_0 \in \mathcal{P}_1$ is such that $\int_{\mathbb{R}_+} x^2 \times Q_0(dx) < \infty$. Assume also that Q_0 is discrete, that is, that its support is contained in $\mathbb{N}^*$. Then uniqueness holds for* (MS), (MP) *and we have uniqueness in law for* (SDE).

In order to use the results of the analysts in the continuous case, we first have to check that for $(X_0, X, \tilde{X}, N)$ a solution to (SDE), $\mathcal{L}(X_t)$ is really a modified solution to (SC): we have to prove that if Q_0 has a density, then for all $t \geq 0$, the law of X_t admits a density.

PROPOSITION 5.3. *Assume that $X_0 > 0$ is a random variable whose law Q_0 is such that $\mathbb{E}(X_0^2) < \infty$. Assume* ($\mathrm{H}_\beta$) *and consider the corresponding T_0. Assume also that Q_0 is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}_+$ and that $K(x, y)$ is nondecreasing* (*e.g., in x when y is fixed*).

Consider a solution $(X_0, X, \tilde{X}, N)$ to (SDE). *Then for all $t \in [0, T_0)$, the law of X_t is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}_+$. Hence the law of X_t is really a weak solution to* (SC), *in the sense that if $f(x, t)$ denotes the density of X_t, then $n(x, t) = f(x, t)/x$ is a weak solution to* (SC), *in the sense of the Proposition* 2.3.

PROOF. Let us denote by $f_0(x)$ the density of the law of X_0. Let $t \in (0, T_0)$ be fixed. Consider a Lebesgue-null set $\mathcal{A}$. Our aim is to check that $\mathbb{P}(X_t \in \mathcal{A}) = 0$.

First notice that

$$\begin{aligned}\mathbb{P}(X_t \in \mathcal{A}) &= \int_0^\infty \mathbb{P}(X_t \in \mathcal{A} | X_0 = x) f_0(x)\,dx \\ &= \mathbb{E}\left(\int_0^\infty \mathbb{1}_{\mathcal{A}}(X_t^x) f_0(x)\,dx\right)\end{aligned} \tag{5.1}$$

where X^x is a solution, on $[0, T_0)$, of the following standard SDE (here $\tilde{X}$ is known, fixed and behaves as a parameter):

$$X_t^x = x + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \le \frac{K(X_{s-}^x, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, dz). \tag{5.2}$$

We will prove that for almost all ω, the map $x \mapsto X_t^x(\omega)$ can be written as $X_t^x(\omega) = x + \phi_{t,\omega}(x)$, for some increasing function $\phi_{t,\omega}$. This will allow us to conclude, thanks to Lemma A.2 of the Appendix, that for almost all ω,

$$\int_0^\infty \mathbb{1}_{\mathcal{A}}(X_t^x)\,dx = 0 \tag{5.3}$$

so that

$$\int_0^\infty \mathbb{1}_{\mathcal{A}}(X_t^x) f_0(x)\,dx = 0 \tag{5.4}$$

and hence, using (5.1) that $\mathbb{P}(X_t \in \mathcal{A}) = 0$, which is our aim.

It remains to check that for almost all ω, $X_t^x(\omega) = x + \phi_{t,\omega}(x)$, for some increasing function $\phi_{t,\omega}$. It of course, suffices to prove that, for all $x > y$, $X_t^x - X_t^y \ge x - y$.

Let thus $x > y$ be fixed. Consider the following stopping time:

$$\tau = \inf\{s \in [0, T_0) \mid X_s^x < X_s^y\}. \tag{5.5}$$

Then it is clear that for all $t < \tau$, since K is nondecreasing,

$$\begin{aligned}&\int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \le \frac{K(X_{s-}^x, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, dz) \\ &\quad \ge \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \le \frac{K(X_{s-}^y, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds, d\alpha, dz)\end{aligned} \tag{5.6}$$

from which we deduce that for all $s < \tau$,

$$X_s^x - X_s^y \ge x - y. \tag{5.7}$$

It remains to prove that $\tau = T_0$. Let us assume that for some ω, $\tau(\omega) < T_0$. We deduce from (5.7) that

$$X_{\tau-}^x - X_{\tau-}^y \ge x - y. \tag{5.8}$$

Hence, still using the fact that K is nondecreasing, we obtain that, for some random $\alpha_\tau \in [0,1]$, $z_\tau \in [0,\infty)$,

$$\begin{aligned}\Delta X^x_\tau &= \tilde{X}_{\tau-}(\alpha_\tau)\mathbb{1}_{\left\{z_\tau \le \frac{K(X^x_{\tau-},\tilde{X}_{\tau-}(\alpha_\tau))}{\tilde{X}_{\tau-}(\alpha_\tau)}\right\}} \\ &\le \tilde{X}_{\tau-}(\alpha_\tau)\mathbb{1}_{\left\{z_\tau \le \frac{K(X^y_{\tau-},\tilde{X}_{\tau-}(\alpha_\tau))}{\tilde{X}_{\tau-}(\alpha_\tau)}\right\}} = \Delta X^y_\tau.\end{aligned} \tag{5.9}$$

We deduce that

$$X^x_\tau = X^x_{\tau-} + \Delta X^x_\tau \ge x - y + X^y_{\tau-} + \Delta X^y_\tau \ge x - y + X^y_\tau \tag{5.10}$$

which contradicts the definition of τ. □

Thanks to the previous proposition, we are able to state the following uniqueness result:

PROPOSITION 5.4. *Let Q_0 satisfy $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$. Assume* (H$_\beta$) *and consider the corresponding T_0. Assume also that K is nondecreasing and satisfies the regularity condition*: *there exists a locally bounded function ζ on* $[0,\infty)^2$ *such that for all $x, x', y \in \mathbb{R}_+$*,

$$|K(x,y) - K(x',y)| \le |x - x'|\zeta(x,x')(1+y^2). \tag{5.11}$$

Assume also that Q_0 admits a density $f_0(x)$ and set $n_0(x) = f_0(x)/x$. Assume that uniqueness of a weak solution to (SC) *with initial condition n_0 and kernel K holds. Then there exists a unique solution Q to* (MP) *with initial condition Q_0. Thus uniqueness in law holds for* (SDE), *that is*, *any solution $(X_0, X, \tilde{X}, N)$ to* (SDE) *with $\mathcal{L}(X_0) = Q_0$ satisfies $\mathcal{L}(X) = Q$.*

Notice that (5.11) always holds when $K(x,y)$ is of the form $a + b(x+y) + cxy$, for some nonnegative constants a, b, c.

PROOF OF PROPOSITION 5.4. Let Q be a solution to (MP). Thanks to Propositions 5.3 and 3.4, we know that for all t, $Q_t(dx) = f(t,x)\,dx$, for some function $f : [0,T_0) \times \mathbb{R}_+ \to \mathbb{R}_+$. Hence, Proposition 2.3(ii) and Remark 2.6 show that $f(x,t) = xn(x,t)$, where n is the unique solution of (SC). Since $Q_0 \in \mathcal{P}_1$ and $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$, it is easily deduced that for all $T < T_0$,

$$\sup_{t\in[0,T]} \int_0^\infty (x + x^2 + x^3) n(x,t)\,dx = \sup_{t\in[0,T]} \big[1 + \mathbb{E}(X_t) + \mathbb{E}(X_t^2)\big] < \infty. \tag{5.12}$$

Uniqueness of $\{Q_t\}_{t\in[0,T_0)}$ is proved, but we need more: we want to prove uniqueness of $Q \in \mathcal{P}_1^{\uparrow}([0,T_0), \mathcal{H}_{Q_0})$. As Q satisfies (MP) it also satisfies the simple (because linear) martingale problem (MPS): for all $\phi \in C_b^1(\mathbb{R}_+)$,

$$\phi(Z_t) - \phi(Z_0) - \int_0^t \int_{\mathbb{R}_+} \big(\phi(Z_s + y) - \phi(Z_s)\big) K(Z_s, y) n(y,s)\,dy\,ds \tag{5.13}$$

is a Q-martingale, Z standing for the canonical process of $\mathbb{D}^{\uparrow}([0,T_0),\mathcal{H}_{Q_0})$. We will prove the uniqueness for (MPS). In this way, we will deduce that Q is entirely determined, since any solution to (MP) satisfies also (MPS). This will conclude the proof.

But uniqueness for (MPS) is equivalent to the uniqueness in law for the following SDE:

$$Y_t = X_0 + \int_0^t \int_{\mathbb{R}_+} \int_0^\infty y \mathbb{1}_{\{z \le K(Y_{s-},y)/y\}} \mu(ds,dy,dz), \tag{5.14}$$

$\mu(ds,dy,dz)$ being a Poisson measure on $[0,T_0) \times \mathbb{R}_+ \times [0,\infty)$ with intensity measure $ds(yn(y,s)\,dy)\,dz$. Strong uniqueness (which implies the uniqueness in law) holds for this equation, thanks to standard arguments: local Lipschitz continuity and at most linear growth. Indeed, for all $u \ge 0$, all $T < T_0$, we obtain, using (H_β) and (5.12),

$$\begin{aligned}
&\sup_{s\in[0,T]} \int_{\mathbb{R}_+} \int_0^\infty y \mathbb{1}_{\{z \le K(u,y)/y\}}\, dz\, yn(y,s)\, dy \\
&\qquad \le A(1+u) \sup_{s\in[0,T]} \int_{\mathbb{R}_+} (y+y^3) n(y,s)\, dy \\
&\qquad \le A_T(1+u),
\end{aligned} \tag{5.15}$$

the constant A_T depending only on T. We also have, for all u, u' in $[0,\infty)$, all $T < T_0$, by using (5.11) and (5.12), that

$$\begin{aligned}
&\sup_{s\in[0,T]} \int_{\mathbb{R}_+} \int_0^\infty \left| y \mathbb{1}_{\{z \le K(u,y)/y\}} - y \mathbb{1}_{\{z \le K(u',y)/y\}} \right| dz\, yn(y,s)\, dy \\
&\qquad \le \sup_{s\in[0,T]} \int_{\mathbb{R}_+} |K(u,y) - K(u',y)| yn(y,s)\, dy \\
&\qquad \le \zeta(u,u')|u-u'| \sup_{s\in[0,T]} \int_{\mathbb{R}_+} (y+y^3) n(y,s)\, dy \\
&\qquad \le A_T \zeta(u,u')|u-u'|.
\end{aligned} \tag{5.16}$$

Using these properties, the strong uniqueness is easily checked for equation (5.14). This implies the uniqueness for (MPS) and concludes the proof. $\square$

We finally deduce the following corollary from Aldous [1], Principle 1.

COROLLARY 5.5. *Assume that Q_0 belongs to $\mathcal{P}_1$ and that $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$. Assume also that $K(x,y) \le C(1+x+y)$ for some positive constant C, that K is nondecreasing and that the regularity condition* (5.11) *holds.*

In addition, assume that Q_0 admits a density $f_0(x)$ and that $\int_{\mathbb{R}_+} \frac{1}{x} Q_0(dx) < \infty$. Then uniqueness in law holds for (SDE) *and so does uniqueness for* (MP).

We conclude this section by stating a remark in the explicit situation of the multiplicative kernel: we can get rid of the condition $\int_{\mathbb{R}_+} x^2 Q_0(dx) < \infty$ and obtain uniqueness by ourselves.

REMARK 5.6. Assume that $K(x, y) = xy$. Let Q_0 belong to $\mathcal{P}_1$ and $T_0 = 1/\int_{\mathbb{R}_+} tx Q_0(dx)$. Then, one may copy the ideas of Desvillettes, Graham and Méléard [5] and obtain directly from a specific Picard iteration the following existence result: for any random variable X_0 of law Q_0, any independent Poisson measure $N(dt, d\alpha, dz)$ with intensity measure $dt\, d\alpha\, dz$, there exists a solution $(X_0, X, \tilde{X}, N)$ to (SDE) on $[0, T_0)$.

Still following [5], one can prove by using directly probabilistic arguments the following uniqueness result: the law $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ is unique and depends only on Q_0.

Hence, in this very particular case, probabilistic arguments allow to obtain existence and uniqueness for (MP).

6. The nonlinear process as a limit of a Marcus–Lushnikov procedure. The aim of this section is to construct a connection between the Marcus–Lushnikov process [13], [14] and our nonlinear process. For the sake of simplicity, we treat here only the discrete case, but what follows can be extended to the general case without difficulty. The proof is done under the hypothesis $(\mathrm{H}_{1/2})$ and a third order moment for the initial condition. We do not know if the result remains valid under (H_1).

For an initial condition $\mu_0 = \{n_0(k)\}_{k\geq 1}$, denote by $|\mu_0| = \sum_{k\geq 1} n_0(k)$. Assume as usual that $\sum_{k\geq 1} kn_0(k) = 1$ and that $\sum_{k\geq 1} k^4 n_0(k) < \infty$. Under these assumptions uniqueness for (SD) and (MP) is known (see Corollary 5.2). We denote by $\mathcal{M}_+(\mathbb{N}^*)$ the set of finite nonnegative measures on $\mathbb{N}^*$. We first of all introduce an approximation of μ_0.

DEFINITION 6.1. For each $n \in \mathbb{N}^*$ we define a deterministic element of $\mathcal{M}_+(\mathbb{N}^*)$, of the form $\mu_0^n = \frac{1}{m_n}\sum_{i=1}^n \delta_{x_0^{i,n}}$ with $m_n = \sum_{i=1}^n x_0^{i,n}$. Moreover we require that μ_0^n tends to μ_0 as n tends to infinity, in the sense that for every function $\phi : \mathbb{N}^* \mapsto \mathbb{R}_+$ with at most linear growth, $\mu_0^n(\phi)$ tends to $\mu_0(\phi)$.

This has to be thought as a system of n particles labeled by their sizes $x_0^{i,n}$ and m_n is the total mass of the system.

We now recall the construction of a Marcus–Lushnikov process associated with this initial condition and with the coagulation kernel K:

Each pair of particles $\{x_i, x_j\}$ coalesce at rate $K(x_i, x_j)/m_n$ to form a new particle $x_i + x_j$ and so on.

Denote, for each $t > 0$, by $n(t)$ the (random) number of particles at time t, and by $X_t^{1,n}, \dots, X_t^{n(t),n}$ the sizes of these particles. Then consider the Markov process $\mu_t^n = \frac{1}{m_n}\sum_{i=1}^{n(t)} \delta_{X_t^{i,n}}$, which belongs a.s. to $\mathbb{D}([0,\infty), \mathcal{M}_+(\mathbb{N}^*))$. This is the Marcus–Lushnikov process, we refer to Norris [16] for further details.

Since by its definition, μ_0^n goes weakly to μ_0, it is well known (see, e.g., Norris [16]), that $\{\mu_t^n\}_{t\geq 0}$ goes weakly, in $\mathbb{D}([0,\infty), \mathcal{M}_+(\mathbb{N}^*))$, to the (deterministic) solution $\{\mu_t\}_{t\geq 0}$ of the Smoluchowski equation: more precisely $n(k,t) := \mu_t(\{k\})$ satisfies (SD).

We consider now a more precise description of this Marcus–Lushnikov procedure.

Each initial particle $x_0^{i,n}$ can be seen as a cluster composed of monomers $\bar{h}_i$. The aim is to follow the evolution of a fixed monomer so we are led to order these monomers in the following way:

$$\begin{aligned}
x_0^{1,n} &= \{\bar{h}_1, \dots, \bar{h}_{x_0^{1,n}}\},\\
x_0^{2,n} &= \{\bar{h}_{x_0^{1,n}+1}, \dots, \bar{h}_{x_0^{1,n}+x_0^{2,n}}\},\\
&\ \ \vdots\\
x_0^{n,n} &= \{\bar{h}_{m_n - x_0^{n,n}+1}, \dots, \bar{h}_{m_n}\}.
\end{aligned}$$

Then, using a random permutation σ of $\{1,\dots,m_n\}$ we reordinate $h_1 = \bar{h}_{\sigma(1)}, \dots, h_{m_n} = \bar{h}_{\sigma(m_n)}$. This step is purely technical and its only interest is to symmetrize the initial system. Our aim is to prove that in a certain sense, the stochastic process defined as the size of the cluster containing h_1 [which clearly belongs a.s. to $\mathbb{D}^{\uparrow}([0,\infty), \mathbb{N}^*)$], goes in law, as n tends to infinity, to our nonlinear process X, solution to (SDE).

NOTATION 6.2.

1. For all $i \in \{1,\dots,m_n\}$, all $t \geq 0$, we set

$$F_i^n(t) = \bigl\{j \in \{1,\dots,m_n\} \mid h_i \text{ and } h_j \text{ are in the same cluster}\bigr\}. \tag{6.1}$$

   Notice that for each i, $F_i^n(\cdot)$ is nondecreasing in an obvious sense, that if $F_i^n(t) = F_j^n(t)$, then $F_i^n(t+h) = F_j^n(t+h)$ for all $h \geq 0$. Moreover, for each i, j and for each $t \geq 0$, either $F_i^n(t) = F_j^n(t)$ or $F_i^n(t) \cap F_j^n(t) = \varnothing$. Furthermore, $\forall t \geq 0, \cup_i F_i^n(t) = \{1,\dots,m_n\}$.
2. For $F \subset \mathbb{N}^*$, we denote by $|F|$ the cardinal of F. For F and G subsets of $\mathbb{N}^*$, we denote by $F + G := F \cup G$, which will allow some Poissonian notation.
3. We denote by $e_k = (0,\dots,1,\dots,0) \in \mathbb{R}^{m_n}$, the 1 being at the kth place.

Then we may write the evolution of the vector $(F_1^n(t), \dots, F_{m_n}^n(t))$ of subsets of $\mathbb{N}^*$ in terms of Poisson measures.

PROPOSITION 6.3. *One may find a Poisson measure* $N^n(ds, di, dj, dz)$ *on* $[0,\infty) \times \{1, \dots, m_n\}^2 \times [0,\infty)$, *with intensity* $(1/m_n)\,ds \sum_k \delta_k(di) \sum_k \delta_k(dj)\,dz$ *such that*

$$\begin{pmatrix} F_1^n(t) \\ \vdots \\ F_{m_n}^n(t) \end{pmatrix} = \begin{pmatrix} F_1^n(0) \\ \vdots \\ F_{m_n}^n(0) \end{pmatrix} + \int_0^t \int_i \int_j \int_0^\infty \Biggl\{ \sum_{k \in F_j^n(s-) + F_i^n(s-)} \bigl(F_i^n(s-) + F_j^n(s-)\bigr) e_k \Biggr\} \times \mathbb{1}_{\left\{z \le \frac{K(|F_i^n(s-)|, |F_j^n(s-)|)}{2|F_i^n(s-)||F_j^n(s-)|}\right\}} N^n(ds, di, dj, dz). \tag{6.2}$$

The only problem is to understand that the rate of coagulation of $F_i^n(s)$ with $F_j^n(s)$ is $\frac{K(|F_i^n(s)|, |F_j^n(s)|)}{2m_n|F_i^n(s)||F_j^n(s)|}$. This is clear because each pair of "true" particles of sizes x_i, x_j is represented $2x_ix_j$ times from the $\{|F_k^n|\}_{k \in \{1,\dots,m_n\}}$ point of view.

REMARK 6.4. We can reobtain the Marcus–Lushnikov process by writing

$$\mu_t^n = \frac{1}{m_n} \sum_{i=1}^{m_n} \frac{1}{|F_i^n(t)|} \delta_{|F_i^n(t)|}. \tag{6.3}$$

Let us finally state the main result of this section.

THEOREM 6.5. *Assume* $(\mathrm{H}_{1/2})$ *and that the initial condition* $Q_0(dx) = x\mu_0(dx)$ *has a moment of order* 3. *Denote by*

$$Q^n = \frac{1}{m_n} \sum_{i=1}^{m_n} \delta_{|F_i^n(\cdot)|} \tag{6.4}$$

which belongs a.s. to $\mathcal{P}(\mathbb{D}^\uparrow([0,\infty), \mathbb{N}^*))$. *Then*:

(i) Q^n *goes in law, in* $\mathcal{P}(\mathbb{D}^\uparrow([0,\infty), \mathbb{N}^*))$, *to the unique solution* Q *of* (MP) *with initial condition* Q_0.

(ii) *Let* $(X_0, X, \tilde{X}, N)$ *be a solution to* (SDE) *with initial distribution* Q_0. *Then* $|F_1^n|$ *goes in law, in* $\mathbb{D}^\uparrow([0,\infty), \mathbb{N}^*)$, *to* X.

PROOF. First notice that thanks to the symmetry of the particle system, the law $\mathcal{L}(|F_i^n(.)|)$ does not depend on $i \in \{1, \dots, m_n\}$. We now break the proof in several steps.

Step 1. First notice that Q_0^n can be written as $Q_0^n = \frac{1}{m_n} \sum_{i=1}^n x_0^{i,n} \delta_{x_0^{i,n}}$. Hence it is easily checked that Q_0^n goes weakly to Q_0, in the sense that for any bounded function $\phi : \mathbb{N}^* \mapsto \mathbb{R}_+$, $Q_0^n(\phi)$ tends to $Q_0(\phi)$.

Step 2. Using Proposition 6.3, it is easily checked that for any $\phi \in C_b^1(\mathbb{R}_+)$,

$$\begin{aligned} \langle Q_t^n, \phi\rangle = \langle Q_0^n, \phi\rangle \\ &+ \frac{1}{m_n}\int_0^t \int_i \int_j \int_0^\infty \mathbb{1}_{\{F_i^n(s-)\neq F_j^n(s-)\}} \\ &\qquad \times \{(|F_i^n(s-)| + |F_j^n(s-)|)\phi(|F_i^n(s-)| + |F_j^n(s-)|) \\ &\qquad\quad - |F_i^n(s-)|\phi(|F_i^n(s-)|) - |F_j^n(s-)|\phi(|F_j^n(s-)|)\} \\ &\qquad \times \mathbb{1}_{\left\{z \le \frac{K(|F_i^n(s-)|,|F_j^n(s-)|)}{2|F_i^n(s-)||F_j^n(s-)|}\right\}} N^n(ds, di, dj, dz). \end{aligned} \tag{6.5}$$

Apply (6.5) with $\phi(x) = x^3$. By using the nondecreasing property for $|F_i^n|$, hypothesis ($\mathrm{H}_{1/2}$) and Definition 6.1, we obtain that for all T, there exists $C_T < \infty$ such that for any $i \in \{1, \dots, m_n\}$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |F_i^n(t)|^3\Big] = \mathbb{E}\big[|F_i^n(T)|^3\big] = \mathbb{E}\Bigg[\frac{1}{m_n}\sum_{j=1}^{m_n} |F_j^n(T)|^3\Bigg] \le C_T. \tag{6.6}$$

Step 3. We now want to prove that Q^n is tight. It suffices to prove, thanks to the symmetry of the particle system (see Méléard [15], Lemma 4.5), that $\mathcal{L}(|F_1^n(\cdot)|)$ is tight in $\mathbb{D}^\uparrow([0,\infty), \mathbb{N}^*)$. This is easily obtained by using the Aldous criterion [see Theorem A.1 and (6.6)].

Step 4. Under the hypothesis of the theorem, we have uniqueness for (MP) (see Corollary 5.2). In order to conclude the proof of (i), we have to show that the weak limit point $\bar{Q}$ of any converging subsequence $\{Q^{n_k}\}$ satisfies a.s. (MP). To this aim, we proceed as in the proof of Lemma 3.9. We have to prove that for any $g_1, \dots, g_l$ in $C_b(\mathbb{R}_+)$, any $0 \le s_1 < \cdots < s_l < s < t$ and any $\phi \in C_b^1(\mathbb{R}_+)$, $\langle \bar{Q} \otimes \bar{Q}, F\rangle = 0$ a.s., where the map F from $\mathbb{D}^\uparrow([0,\infty), \mathbb{N}^*) \times \mathbb{D}^\uparrow([0,\infty), \mathbb{N}^*)$ into $\mathbb{R}$ is defined by (3.28).

For symmetrical reasons, we have only to check that $\langle \bar{Q} \otimes \bar{Q}, G\rangle = 0$ a.s., where

$$\begin{aligned} G(x,y) = g_1(x(s_1)) \cdots g_l(x(s_l)) \\ &\times \Bigg\{\phi(x(t)) - \phi(x(s)) \\ &\quad - \int_s^t \big[(x(u) + y(u))\phi(x(u) + y(u)) \\ &\qquad - x(u)\phi(x(u)) - y(u)\phi(y(u))\big]\frac{K(x(u), y(u))}{2x(u)y(u)}\,du\Bigg\}. \end{aligned} \tag{6.7}$$

Although G is not really bounded nor continuous, one can prove, using the same kind of arguments as in the proof of Lemma 3.9, that

$$\mathbb{E}\big[|\langle \bar{Q} \otimes \bar{Q}, G\rangle|\big] = \lim_k \mathbb{E}\big[|\langle Q^{n_k} \otimes Q^{n_k}, G\rangle|\big]. \tag{6.8}$$

Hence we just need to prove that, for $U_n = \langle Q^n \otimes Q^n, G\rangle$, $\mathbb{E}(|U_n|)$ goes to 0. An easy computation using (6.5) shows that

$$U_n = \frac{1}{m_n} \sum_{i=1}^{m_n} g_1(|F_i^n(s_1)|) \cdots g_l(|F_i^n(s_l)|)\big[M_i^{n,\phi}(t) - M_i^{n,\phi}(s) + P_i^{n,\phi}(s,t)\big] \tag{6.9}$$

with

$$\begin{aligned} M_i^{n,\phi}&(t) \\ &= \int_0^t \int_j \int_0^\infty \mathbb{1}_{\{F_i^n(s-)\neq F_j^n(s-)\}} \\ &\qquad \times \big\{\big(|F_i^n(s-)| + |F_j^n(s-)|\big)\phi\big(|F_i^n(s-)| + |F_j^n(s-)|\big) \\ &\qquad\quad - |F_i^n(s-)|\phi\big(|F_i^n(s-)|\big) - |F_j^n(s-)|\phi\big(|F_j^n(s-)|\big)\big\} \\ &\qquad \times \mathbb{1}_{\left\{z \le \frac{K(|F_i^n(s-)|,|F_j^n(s-)|)}{2|F_i^n(s-)||F_j^n(s-)|}\right\}} \tilde{N}^n(ds, \{i\}, dj, dz) \end{aligned} \tag{6.10}$$

where $\tilde{N}^n(ds, \{i\}, dj, dz) = N^n(ds, \{i\}, dj, dz) - \frac{1}{m_n} ds \sum_{k=1}^{m_n} \delta_k(dj)\, dz$ and where

$$\begin{aligned} P_i^{n,\phi}&(s,t) \\ &= \frac{1}{m_n}\int_s^t \sum_{j=1}^{m_n} \mathbb{1}_{\{F_i^n(u)=F_j^n(u)\}} \big\{\big(|F_i^n(u)| + |F_j^n(u)|\big)\phi\big(|F_i^n(u)| + |F_j^n(u)|\big) \\ &\qquad\qquad - |F_i^n(u)|\phi\big(|F_i^n(u)|\big) - |F_j^n(u)|\phi\big(|F_j^n(u)|\big)\big\} \\ &\qquad\qquad \times \frac{K(|F_i^n(u)|, |F_j^n(u)|)}{2|F_i^n(u)||F_j^n(u)|}\, du. \end{aligned} \tag{6.11}$$

Since for $i \neq i'$, the Poisson measures $N^n(ds, \{i\}, dj, dz)$ and $N^n(ds, \{i'\}, dj, dz)$ are independent, the martingales $M_i^{n,\phi}$ are orthogonal so that

$$\forall\, i \neq i', \qquad \langle M_i^{n,\phi}, M_{i'}^{n,\phi}\rangle \equiv 0. \tag{6.12}$$

One easily checks, using $(\mathrm{H}_{1/2})$, that for any i and any T,

$$\langle M_i^{n,\phi}\rangle_t \le \frac{A}{m_n}\int_0^t \sum_{j=1}^{m_n} \big[|F_i^n(u)|^2|F_j^n(u)| + |F_i^n(u)||F_j^n(u)|^2\big]\, du \tag{6.13}$$

the constant A depending only on the coagulation kernel K and on the test function ϕ. Similarly, an easy computation using $(\mathrm{H}_{1/2})$ shows that for some constant A,

$$
\begin{aligned}
|P_i^{n,\phi}(s,t)| &\le \frac{A}{m_n}\int_0^t \sum_{j=1}^{m_n} \mathbb{1}_{\{F_i^n(u)=F_j^n(u)\}}\big[|F_i^n(u)|+|F_j^n(u)|\big]\,du \\
&\le \frac{2A}{m_n}\int_0^t |F_i^n(u)|^2\,du.
\end{aligned}
\tag{6.14}
$$

Indeed, the number of j's such that $F_i^n(u)=F_j^n(u)$ is exactly equal to $|F_i^n(u)|$.

Using (6.12), (6.13), (6.14) and then (6.6), we obtain the existence of some constant A_t, depending only on g_i, ϕ, K and on the (fixed) instant t, such that

$$
\begin{aligned}
&\mathbb{E}[|U_n|] \\
&\quad\le \frac{A_t}{m_n}\sum_{i=1}^{m_n}\mathbb{E}\big[|P_i^{n,\phi}(s,t)|\big] \\
&\quad + \left[\mathbb{E}\left(\left\{\frac{1}{m_n}\sum_{i=1}^{m_n} g_1\big(|F_i^n(s_1)|\big)\cdots g_l\big(|F_i^n(s_l)|\big)\big[M_i^{n,\phi}(t)-M_i^{n,\phi}(s)\big]\right\}^2\right)\right]^{1/2} \\
&\quad\le \frac{A_t}{m_n}\sum_{i=1}^{m_n}\mathbb{E}\big[|P_i^{n,\phi}(s,t)|\big] + A_t\left[\mathbb{E}\left\{\frac{1}{m_n^2}\sum_{i=1}^{m_n}\langle M_i^{n,\phi}\rangle_t\right\}\right]^{1/2} \\
&\quad\le A_t/\sqrt{m_n}
\end{aligned}
\tag{6.15}
$$

which goes to 0 as n tends to infinity. This concludes the proof of (i).

Step 5. We finally deduce (ii). We just have to prove that, for Q a solution to (MP) [which is the law of X, for $(X_0, X, \tilde{X}, N)$ a solution to (SDE)], for all ϕ continuous and bounded from $\mathbb{D}^{\uparrow}([0,\infty),\mathbb{N}^*)$ into $\mathbb{R}$,

$$
\lim_n \mathbb{E}\big[\phi(|F_1^n(\cdot)|)\big] = \langle Q,\phi\rangle.
\tag{6.16}
$$

This is obvious from (i), since for symmetrical reasons,

$$
\mathbb{E}\big[\phi(|F_1^n(\cdot)|)\big] = \mathbb{E}[\langle Q^n,\phi\rangle],
\tag{6.17}
$$

and since the map $\nu \mapsto \langle \nu,\phi\rangle$ is continuous and bounded from $\mathcal{P}(\mathbb{D}^{\uparrow}([0,\infty),\mathbb{N}^*))$ into $\mathbb{R}$. The proof of the theorem is now complete. $\square$

APPENDIX

First, we recall the Aldous criterion for tightness (see Jacod and Shiryaev [11], page 320).

THEOREM A.1. *Let $\{X_t^n\}_{t\in[0,T_0)}$ be a family of càdlàg adapted processes on $[0,T_0)$, for some $T_0\le\infty$. Denote by $Q^n\in\mathcal{P}(\mathbb{D}([0,T_0),\mathbb{R}))$ the law of X^n. Assume that*:

(i) *for all* $T < T_0$, $\sup_n \mathbb{E}[\sup_{t\in[0,T]} |X_t^n|] < \infty$;
(ii) *for all* $T < T_0$, *all* $\eta > 0$,

$$\sup_n \sup_{(S,S')\in U_T(\delta)} P\big[|X_{S'}^n - X_S^n| \geq \eta\big] \underset{\delta\to 0}{\longrightarrow} 0, \tag{A.1}$$

where $U_T(\delta)$ *denotes the set of couples* (S, S') *of stopping times satisfying a.s.* $0 \leq S \leq S' \leq (S+\delta)\wedge T$.

Then the family $\{Q^n\}$ *is tight. Furthermore, any limiting point* Q *of this family is the law of a quasi-left continuous process, that is, for all* $t \in [0, T_0)$ *fixed,*

$$\int_{\mathbb{D}([0,T_0),\mathbb{R})} \mathbb{1}_{\{\Delta x(t)\neq 0\}} Q(dx) = 0. \tag{A.2}$$

We now state an easy absolute continuity result.

LEMMA A.2. *Let* φ *be an increasing map from* $\mathbb{R}_+$ *into itself. Let* $\mathcal{A}$ *be a Lebesgue-null subset of* $\mathbb{R}_+$. *Then*

$$\int_0^\infty \mathbb{1}_{\mathcal{A}}\big(x+\varphi(x)\big)\,dx = 0. \tag{A.3}$$

We carry on with a generalized Gronwall lemma (see Beesack [3], page 6).

LEMMA A.3. *Let* $a, b \geq 0$. *Consider a continuous function* g *on* $[0, T]$, *satisfying for all* $t \in [0, T]$,

$$g(t) \leq a + b\int_0^t g^2(s)\,ds. \tag{A.4}$$

Then, for all $t < T_0 = 1/ab$,

$$g(t) \leq \frac{a}{1-abt}. \tag{A.5}$$

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